Homework Solutions 7b (Schroeder chapter 7)

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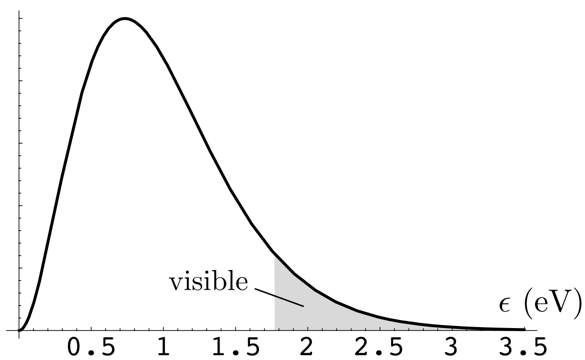
(a) The power $\mathcal{P} = e\sigma A T^4$, so

$$A = \frac{\mathcal{P}}{e\sigma T^4} = 6.6 \times 10^{-5}\ \mathrm{m}^2$$

(b) The spectrum has a peak at $E = 2.82\,kT = 0.73$ eV. The wavelength is $\lambda = hc/E = 1.7 \times 10^{-6}$ m, in the infrared.

(c) Plotting, we see that most of of the energy is radiated away in the infrared.



(d) Change variables to $x = E/kT$. At 3000 K, $kT = 0.26$ eV. The visible spectrum is between $1.77/0.26 = 6.8$ and $3.1/0.26 = 11.9$. The energy in the visible is

$$\int_{6.8}^{11.9} dx\, \frac{x^3}{e^x - 1}$$

This needs to be evaluated numerically. All the energy is

$$\int_0^{\infty} dx\, \frac{x^3}{e^x - 1} = \frac{\pi^4}{15}$$

Their ratio is the visible energy fraction, which is 0.083. Not very efficient.

(e) Increasing efficiency requires shifting the spectrum peak toward the visible, which means a higher temperature.

(f) The efficiency requires figuring out the integral ratio in (d) as a function of T, and either numerically differentiating it for a maximum, or plotting it and visually estimating the peak, around 7000 K.

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(a) Putting together the equations provided with the spectrum peak being at $E = 2.82\,kT$ results in $\lambda = 8.4 \times 10^4$ m.

(b) Total power for a blackbody:

$$\mathcal{P} = \sigma A T^4 = 9 \times 10^{-29}\ \mathrm{W} = 6 \times 10^{-10}\ \mathrm{eV/s}$$

This is really low.

(c) The energy lost from the black hole must come from a decrease in mass: $\mathcal{P} = -d(Mc^2)/dt$. Therefore

$$\frac{dM}{dt} = -\frac{\sigma A T^4}{c^2} = -\frac{H}{M^2} \quad \text{with} \quad H = 4.0 \times 10^{15}\ \mathrm{kg^3/s}$$

This gives $M^2 dM = -H dt$, which we can integrate to get the lifetime τ:

$$\tau = \int_0^\tau dt = -\frac{1}{H}\int_{M_i}^0 dM\, M^2 = \frac{M_i^3}{3H}$$

(d) For a solar mass, $\tau = 7 \times 10^{74}$ s; about 10^{57} times the age of the universe.

(e) The age of the universe is about 4×10^{17} s. The initial mass of a black hole with that lifetime would be 1.7×10^{11} kg, smaller than the sun by about a factor of 10^{19}. The peak of its spectrum would be at about $\lambda = 7 \times 10^{-15}$ m, an extremely energetic gamma ray.

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(a) The number of magnons is

$$N_m = \sum_{n_x, n_y, n_z} \frac{1}{e^{E/kT} - 1}$$

As usual, $E = \hbar^2 k^2/2m^* = h^2 n^2/8m^* L^2$. Again, as usual, convert the sum to an integral using spherical coordinates, ending up with

$$N_m = \frac{\pi}{2}\int_0^\infty dn\, \frac{n^2}{e^{n^2h^2/8m^*L^2kT} - 1}$$

As usual, change variables to $x = E/kT$, and after some algebra, you get

$$N_m = 2\pi V\left(\frac{2m^*kT}{h^2}\right)^{3/2}\int_0^\infty dx\,\frac{x^{1/2}}{e^x - 1} = 2\pi V\left(\frac{2m^*kT}{h^2}\right)^{3/2}(2.315)$$

(b) Each magnon reduces the magnetization by $2\mu_B$. Therefore the fractional reduction in magnetization is

$$\frac{N_m}{N} = \left(\frac{T}{T_0}\right)^{3/2} \quad \text{where} \quad T_0 = (0.0839)\frac{h^2}{m^*k}\left(\frac{N}{V}\right)^{2/3}$$

For iron, $T_0 = 4150$ K.

(c) The energy is

$$\begin{aligned}
U &\approx \frac{\pi}{2}\int_0^\infty dn\, n^2 \frac{1}{e^{n^2h^2/8m^*L^2kT} - 1}\frac{h^2n^2}{8m^*L^2} \\
&= 2\pi V\left(\frac{2m^*kT}{h^2}\right)^{3/2}(kT)\int_0^\infty dx\,\frac{x^{3/2}}{e^x - 1} \\
&= 2\pi V\left(\frac{2m^*kT}{h^2}\right)^{3/2} kT\,(1.783) = (31.69)\,kV\left(\frac{km^*}{h^2}\right)^{3/2} T^{5/2}
\end{aligned}$$

$$C_V = \frac{\partial U}{\partial T} = (31.69)\frac{5}{2}\,kV\left(\frac{km^*T}{h^2}\right)^{3/2}$$

Therefore

$$\frac{C_V}{Nk} = \left(\frac{T}{T_1}\right)^{3/2} \quad \text{with} \quad T_1 = \frac{h^2}{km^*}\left(\frac{N}{V}\right)^{2/3}\left(\frac{2}{5(31.69)}\right)^{2/3} = 0.646\,T_0$$

For iron, this is $T_1 = 2680$ K, which means the magnon contribution is small at room temperatures and below. But when the temperature is low enough, the magnons will contribute more than the phonons, since the phonon $C_V \propto T^3$.

(d) The only difference here is that the angular integral is over a quarter-circle, and the radial integral involves $dn\,n$ rather than $dn\,n^2$. Aftr changing variables to $x \propto n^2$ (since $E \propto n^2$), this leaves us with an integral like

$$N_m \propto \int_0^\infty dx\,\frac{1}{e^x - 1} = \infty$$

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(a) The ground state has

$$E_0 = \frac{h^2}{8mL^2}(1^2 + 1^2 + 1^2) = 1.14 \times 10^{-32}\ \mathrm{J} = 7.1 \times 10^{-14}\ \mathrm{eV}$$

(b) Using equation 7.126, $kT_c = (0.224)N^{2/3}E_0$. In that case,

$$\frac{kT_c}{E_0} = (0.224)(10000)^{2/3} = 104 \quad \Rightarrow \quad T_c = 8.6 \times 10^{-8}\ \mathrm{K}$$

(c) At $T = 0.9T_c$,

$$N_0 = \left[1 - \left(\frac{T}{T_c}\right)^{3/2}\right] N = 0.146\,N = 1460$$

Then, with equation 7.120, we get $E_0 - \mu = kT/N_0 = 4.6 \times 10^{-15}$ eV. The first excited state have energies of $2E_0$ each. The occupancy of each is

$$N_1 = \frac{1}{e^{(E_1-\mu)/kT} - 1} = 87$$

(d) With 10^6 atoms, T_c will be higher by a factor of $(10^6/10^4)^{2/3} = 21.5$, so $T_c = 1.85 \times 10^{-6}$ K. The fraction in the ground state remains the same, but the actual number is large by a factor of 100. Therefore $E_0 - \mu = 9.8 \times 10^{-16}$ eV, and $N_1 = 1990$. It looks like, if T/T_c is kept the same, N_0/N_1 increases with N.