Homework Solutions 1 (Griffiths Chapter 1)

In the following, I’ll have the solid angle element $d\Omega = \sin\theta \, d\theta \, d\phi$. And therefore, standing alone, $\oint d\Omega = 4\pi$, integrated over the full solid angle.

39 We want to check $\int_V dv \, \nabla \cdot \mathbf{v} = \oint_{\partial V} d\mathbf{a} \cdot \mathbf{v}$. For our spherical volume, $d\mathbf{a} = d\Omega R^2 \hat{r}$.

$$
\int_V dv \, \nabla \cdot \mathbf{v}_1 = \int_V dv \, \frac{1}{r^2} \frac{\partial}{\partial r} r^4 = \oint d\Omega \int_0^R dr \, r^2 (4r) = 4\pi R^4
$$
$$
\oint_{\partial V} d\mathbf{a} \cdot \mathbf{v}_1 = \oint d\Omega \, R^4 = 4\pi R^4
$$

These are obviously equal.

Now, it may seem that

$$
\int_V dv \, \nabla \cdot \mathbf{v}_2 = \int_V dv \, \frac{1}{r^2} \frac{\partial}{\partial r} 1 = 0
$$

but this is misleading, since there are infinities involved when $r \to 0$. Looking at the surface integral

$$
\oint_{\partial V} d\mathbf{a} \cdot \mathbf{v}_2 = \oint d\Omega \, 1 = 4\pi
$$

Since this integral gives $4\pi$ for all $R > 0$, this must mean that $\nabla \cdot \mathbf{v}_2 = 0$ for all $r > 0$, but when we include $r = 0$, the integral is $4\pi$. Those properties define the three-dimensional delta function, so

$$
\nabla \cdot \left( \frac{\hat{r}}{r^2} \right) = \delta^3(\mathbf{r})
$$

45 Change variables in each case:

(a) With $u = 3x$ and $du = 3dx$, the integral becomes

$$
\int_{-6}^6 \frac{du}{3} \delta(u) \left( \frac{2}{3}u + 3 \right) = 1
$$

(b) $u = 1 - x$, $du = -dx$

$$
\int_{-1}^1 du \, \delta(u) \left[ (1-u)^3 + 3(1-u) + 2 \right] = 6
$$
(c) \( u = 3x + 1, \; du = 3dx \)

\[
\int_{-1}^{4} \frac{du}{3} \delta(u) 9 \left( \frac{u - 1}{3} \right)^2 = \frac{1}{3}
\]

(d) \( \int_{-\infty}^{a} dx \delta(x - b) = 0 \) if \( b > a \) and = 1 if \( b < a \). The result is ambiguous if \( b = a \).

49 First, use the delta function result from problem (39):

\[
\int_{V} dv e^{-r} \left[ \nabla \cdot \left( \frac{\hat{r}}{r^2} \right) \right] = \int_{V} dv e^{-r} \delta^3(\mathbf{r}) = 4\pi
\]

Then, integrating by parts,

\[
\int_{V} dv e^{-r} \left[ \nabla \cdot \left( \frac{\hat{r}}{r^2} \right) \right] = \oint_{\partial V} d\mathbf{a} \cdot \left( \frac{\hat{r}}{r^2} \right) e^{-r} - \int_{V} dv \left( \frac{\hat{r}}{r^2} \right) \cdot \nabla(e^{-r}) = \int d\Omega e^{-R} + \int d\Omega \int_{0}^{R} r^2 dr \frac{1}{r^2} e^{-r} = 4\pi e^{-R} - 4\pi e^{-R} + 4\pi = 4\pi
\]

50 Divergences and curls:

\[
\nabla \cdot \mathbf{F}_1 = 0 \quad \nabla \times \mathbf{F}_1 = -\frac{\partial}{\partial x} F_{1z}\hat{y} = -2x\hat{y}
\]

\[
\nabla \cdot \mathbf{F}_2 = 3 \quad \nabla \times \mathbf{F}_2 = 0
\]

Since \( \nabla \times \nabla \phi = 0 \), we can have \( \mathbf{F}_2 = \nabla \phi \), where \( \phi = \frac{1}{2}(x^2 + y^2 + z^2) \) is one possibility for \( \phi \).

Since \( \nabla \cdot \nabla \times \mathbf{A} = 0 \), we can have \( \mathbf{F}_1 = \nabla \times \mathbf{A} \), where \( \mathbf{A} = \frac{1}{3} x^3 \hat{y} \) is one possibility for \( \mathbf{A} \).

Then, note that \( \nabla \cdot \mathbf{F}_3 = \nabla \times \mathbf{F}_3 = 0 \). Therefore we can write

\[
\mathbf{F}_3 = \nabla \phi, \text{ with } \phi = xyz
\]

\[
\mathbf{F}_3 = \nabla \times \mathbf{A}, \text{ with } \mathbf{A} = \frac{1}{4} \left[ x(y^2 - z^2)\hat{x} + y(z^2 - x^2)\hat{y} + z(x^2 - y^2)\hat{z} \right]
\]

63 First, the general case:

\[
\nabla \cdot (r^n \hat{r}) = \frac{1}{r^2} \frac{\partial}{\partial r} (r^{n+2}) = (n + 2)r^{n-1} \text{ for } n > -2
\]
For \( n = -1 \), this gives \( \nabla \cdot \mathbf{v} = 1/r^2 \). Integrating this over a sphere centered on the origin,

\[
\int_V dV \nabla \cdot \mathbf{v} = \oint d\Omega \int_0^R dr = 4\pi R
\]

Using the divergence theorem,

\[
\oint_{\partial V} d\mathbf{a} \cdot \frac{\mathbf{r}}{r} = \oint d\Omega R = 4\pi R
\]

These are the same, so no \( \delta \)-functions are involved.

Using the spherical form for the curl,

\[
\nabla \times (r^n \hat{r}) = 0
\]

since all the \( \theta \) and \( \phi \) components of the vector field are zero, and the \( r \)-component does not depend on \( \theta \) or \( \phi \). Using the result of problem (61b),

\[
\oint_{\partial V} d\mathbf{a} \times (r^n \hat{r}) = \oint_{\partial V} da r^n (\hat{r} \times \hat{r}) = 0
\]