Homework Solutions 2 (Griffiths Chapter 2)

16 With l the length of a cylindrical Gaussian surface, for $s < a$,

$$E(2\pi sl) = \frac{1}{\epsilon_0}\pi s^2 l\rho \quad \Rightarrow \quad E = \frac{\rho s}{2\epsilon_0}$$

For $b > s > a$,

$$E(2\pi sl) = \frac{1}{\epsilon_0}\pi a^2 l\rho \quad \Rightarrow \quad E = \frac{\rho a^2}{2s\epsilon_0}$$

For $s > b$,

$$E(2\pi sl) = 0 \quad \Rightarrow \quad E = 0$$

The direction of $\mathbf{E}$ is $\hat{\mathbf{s}}$ in all cases.

21 With Gauss's law, the electric field is, for $r < R$,

$$E(4\pi r^2) = \frac{1}{\epsilon_0}\frac{4}{3}\pi r^3 \frac{q}{\frac{4}{3}\pi R^3} \quad \Rightarrow \quad \mathbf{E} = \frac{q}{4\pi\epsilon_0}\frac{r}{R^3}\,\hat{\mathbf{r}}$$

For $r \geq R$,

$$E(4\pi r^2) = \frac{1}{\epsilon_0}q \quad \Rightarrow \quad \mathbf{E} = \frac{q}{4\pi\epsilon_0}\frac{1}{r^2}\,\hat{\mathbf{r}}$$

The potential is then, for $r \geq R$, the point particle result of $V(r) = 1/4\pi\epsilon_0 r$. Inside, with $r < R$, there's an extra integral:

$$V(r) = \frac{q}{4\pi\epsilon_0 R} - \frac{q}{4\pi\epsilon_0 R^3}\int_R^r dr'\, r' = \frac{q}{8\pi\epsilon_0 R}\left(3 - \frac{r^2}{R^2}\right)$$

The gradient outside gives the same result as the point particle; inside, we have

$$-\frac{\partial V}{\partial r}\,\hat{\mathbf{r}} = \frac{q}{4\pi\epsilon_0}\frac{r}{R^3}\,\hat{\mathbf{r}}$$

as it should be.

28 Proceed as in example 2.8, positioning the point where we will calculate V on the z-axis. Since the charge density is constant,

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int dv'\,\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} = \frac{\rho}{4\pi\epsilon_0}\int dv'\,\frac{1}{|\mathbf{r}-\mathbf{r}'|}$$

with $|\mathbf{r}-\mathbf{r}'| = r'^2 + z^2 - 2r'z\cos\theta'$. Continuing,

$$\begin{aligned}
V(\mathbf{r}) &= \frac{\rho}{4\pi\epsilon_0} 2\pi \int_0^\pi d\theta' \, \sin\theta' \int_0^R dr' \, r'^2 \frac{1}{r'^2 + z^2 - 2r'z\cos\theta'} \\
&= \frac{\rho}{2\epsilon_0} \int_0^R dr' \, r'^2 \left[\frac{1}{r'z} \sqrt{r'^2 + z^2 - 2r'z\cos\theta'} \right]_0^\pi \\
&= \frac{\rho}{2\epsilon_0 z} \int_0^R dr' \, r' \left[\sqrt{(r'+z)^2} - \sqrt{(r'-z)^2} \right]
\end{aligned}$$

For $z > R$, $\sqrt{(r'-z)^2} = z - r'$, and therefore

$$V(\mathbf{r}) = \frac{\rho}{2\epsilon_0 z} \int_0^R dr' \, r' \left[r' + z + r' - z \right] = \frac{\rho R^3}{3\epsilon_0 z} = \frac{q}{4\pi\epsilon_0 r}$$

as expected.

Inside, we need to split the integral into two: from 0 to z, where $\sqrt{(r'-z)^2} = z - r'$, and from z to R, where $\sqrt{(r'-z)^2} = r' - z$:

$$\begin{aligned}
V(\mathbf{r}) &= \frac{\rho}{2\epsilon_0 z} \left[\int_0^z dr' \, r' \left[r' + z + r' - z \right] + \int_z^R dr' \, r' \left[r' + z - r' + z \right] \right] \\
&= \frac{\rho}{2\epsilon_0 z} \left[\frac{2}{3} z^3 + (zR^2 - z^3) \right] = \frac{\rho}{6\epsilon_0} \left(3R^2 - z^2 \right) = \frac{q}{8\pi\epsilon_0 R} \left(3 - \frac{r^2}{R^2} \right)
\end{aligned}$$

again, as expected.

38 At $r = R$, the charge q will be distributed over the surface, so at the surface $\sigma = q/4\pi R^2$. Since for $a < r < b$, $E = 0$, the enclosed charge must be zero, therefore at $r = a$, $\sigma = -q/4\pi a^2$. And since the outer shell started out electrically neutral, a charge q must be distributed at $r = b$: $\sigma = q/4\pi b^2$.

Using Gauss's law, $E = q/4\pi\epsilon_0 r^2$ for $r > b$ and $R < r < a$. Therefore, at the center,

$$V = -\frac{q}{4\pi\epsilon_0} \left(\int_\infty^b dr \, \frac{1}{r^2} + \int_a^R dr \, \frac{1}{r^2} \right) = \frac{q}{4\pi\epsilon_0} \left(\frac{1}{b} - \frac{1}{a} + \frac{1}{R} \right)$$

When V is set to zero on the shell, the electric field outside also becomes zero. Therefore, since $E_\perp = 0$ at $r = b$, $\sigma = 0$ at $r = b$. To keep $E = 0$ in the interior of the shell, the charge density remains $\sigma = -q/4\pi a^2$ at $r = a$. And since no charge has been drained from the sphere at the center, $\sigma = q/4\pi R^2$ at $r = R$. The voltage at the center will become

$$V = -\frac{q}{4\pi\epsilon_0}\left(\int_a^R dr\,\frac{1}{r^2}\right) = \frac{q}{4\pi\epsilon_0}\left(-\frac{1}{a}+\frac{1}{R}\right)$$

43 Using a cylinder with length l as a Gaussian surface, and with a charge Q over a length l on the inner cable,

$$(2\pi sl)E = \frac{Q}{\epsilon_0} \quad \Rightarrow \quad \mathbf{E} = \hat{\mathbf{s}}$$

The potential difference between the cables is

$$V = \int_a^b ds\,E = \frac{1}{2\pi\epsilon_0}\int_a^b \frac{ds}{s} = \frac{1}{2\pi\epsilon_0}(Q/l)\ln\frac{b}{a}$$

Therefore, the capacitance per length is

$$\frac{C}{l} = \frac{Q/l}{V} = \frac{2\pi\epsilon_0}{\ln\frac{b}{a}}$$

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(a) Generalizing to a charge distribution ρ is straightforward, since E remains linear in q and superposition holds: for multiple charges, we add their electric fields together. A general ρ is also a sum of point charges: $\rho(\mathbf{r}) = \int dv'\,\rho(\mathbf{r}')\delta^3(\mathbf{r}-\mathbf{r}')$. Therefore

$$\mathbf{E}(\mathbf{r}) = \int dv'\,\rho(\mathbf{r}')\frac{1}{4\pi\epsilon_0}\frac{1}{|\mathbf{r}-\mathbf{r}'|^2}\left(1+\frac{|\mathbf{r}-\mathbf{r}'|}{\lambda}\right)e^{-|\mathbf{r}-\mathbf{r}'|/\lambda}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}$$

(b) Since a point charge $\mathbf{E}$ is radially outward, $\nabla\times\mathbf{E} = 0$. The general $\mathbf{E}$ was a sum of point charge fields, so its curl must still be zero. Therefore, there must be a scalar potential such that $\mathbf{E} = -\nabla V$.

(c) Doing the integral (changing variables to $x = r/\lambda$ will help):

$$V(r) = \frac{q}{4\pi\epsilon_0}\int_r^\infty dr'\,\frac{1}{r'^2}\left(1+\frac{r}{\lambda}\right)e^{-r/\lambda} = \frac{q}{4\pi\epsilon_0}\frac{e^{-r/\lambda}}{r}$$

(d) We have, with a spherical volume,

$$\oint_{\partial\mathcal{V}} d\mathbf{a}\cdot\mathbf{E} = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{R^2}\left(1+\frac{R}{\lambda}\right)e^{-R/\lambda}\right]4\pi R^2 = \frac{q}{\epsilon_0}\left[\left(1+\frac{R}{\lambda}\right)e^{-R/\lambda}\right]$$

The voltage integral is

$$\frac{1}{\lambda^2}\int_{\mathcal{V}} dv\, V = \frac{1}{\lambda^2}\frac{q}{4\pi\epsilon_0}\int d\Omega \int_0^R dr\, r^2 \frac{e^{-r/\lambda}}{r} = \frac{q}{\epsilon_0}\left[1-\left(1+\frac{R}{\lambda}\right)e^{-R/\lambda}\right]$$

Adding these, we get q/ϵ_0.

(e) The result generalizes, again because E remains linear in q and superposition holds; $\rho(\mathbf{r}) = \int dv'\,\rho(\mathbf{r}')\delta^3(\mathbf{r}-\mathbf{r}')$.

(f) The bottom of the triangle is still $\mathbf{E} = -\nabla V$; $V = -\int d\mathbf{l}\cdot\mathbf{E}$.

On the right, we have the result of (a), $\nabla\times\mathbf{E} = 0$, and the differential form of (e):

$$\nabla\cdot\mathbf{E} + \frac{V}{\lambda^2} = \frac{\rho}{\epsilon_0}$$

On the left, we have

$$-\nabla^2 V + \frac{V}{\lambda^2} = \frac{\rho}{\epsilon_0} \quad \text{and} \quad V = \frac{1}{4\pi\epsilon_0}\int dv'\,\frac{\rho(\mathbf{r}')e^{-|\mathbf{r}-\mathbf{r}'|/\lambda}}{|\mathbf{r}-\mathbf{r}'|}$$

(g) In the result of (e), we can see that when $\mathbf{E} = 0$, the V-integral remains the same. Since $V = -\int d\mathbf{l}\cdot\mathbf{E}$, V is a constant inside the conductor. Therefore, the V-integral is proportional to the volume, and therefore so must Q_{enc} be. This means a constant charge density ρ.