Homework Solutions 3 (Griffiths Chapter 3)

11 Call \((a, b)\) the location \(r_1\). Put image charges \(-q\) at \((-a, b)\), \((a, b)\) (locations \(r_2\) and \(r_3\)) and \(q\) at \((-a, -b)\) (location \(r_4\)). The potential is then

\[
V(r) = \frac{q}{4\pi\epsilon_0} \left( \frac{1}{|r-r_1|} - \frac{1}{|r-r_2|} - \frac{1}{|r-r_3|} + \frac{1}{|r-r_4|} \right)
\]

The force on the real charge is,

\[
F(r) = \frac{q^2}{4\pi\epsilon_0} \left( -\frac{1}{4b^2}\hat{y} - \frac{1}{4a^2}\hat{x} + \frac{1}{4(a^2+b^2)^{3/2}}(2a\hat{x} + 2b\hat{y}) \right)
\]

The work is, bringing the charge in from infinity along the \(y = b\) line,

\[
W = -\frac{q^2}{4\pi\epsilon_0} \int_a^\infty dx \left( -\frac{x-a}{4b^2 + (x-a)^2} \right) - \frac{1}{(x+a)^2}
\]

\[
= \frac{q^2}{4\pi\epsilon_0} \left( 1 - \frac{2a}{2b} - \frac{1}{2\sqrt{a^2+b^2}} \right)
\]

If we were able to arrange the charge and image charges into a hexagon, octagon, and so forth, with alternating signs for the charges, we would also get \(V = 0\) along planes. So any angle of \(360\degree/2n = \pi/n\) would work, for \(n = 1, 2, 3, \ldots\)

15 It’s most convenient to use the hyperbolic solutions for basis functions along the \(x\)-axis:

\[
V_k(x, y) = (A_k \sinh kx + B_k \cosh kx)(C_k \sin ky + D_k \cos ky)
\]

Putting in the boundary conditions at \(x = 0\) and \(y = 0\), this means \(B_k = 0\) and \(D_k = 0\). Having \(V = 0\) for \(y = a\) requires \(k = n\pi/a\). The general solution giving \(V = V_0(y)\) at \(x = b\) is therefore

\[
V(x, y) = \sum_{n=1}^{\infty} A_n \sinh \left( \frac{n\pi}{a} x \right) \sin \left( \frac{n\pi}{a} y \right)
\]
Since
\[ V_0(y) = \sum_{n=1}^{\infty} A_n \sinh \left( \frac{n\pi}{a} b \right) \sin \left( \frac{n\pi}{a} y \right) \]
this means that \( A_n \) can be found using the orthogonality relationship between sine functions. Operating on each side with \( \frac{2}{a} \int_0^a dy \sin \left( \frac{m\pi}{a} y \right) \) results in
\[ A_m = \frac{2}{a} \sinh \left( \frac{m\pi}{a} b \right) \int_0^a dy \sin \left( \frac{m\pi}{a} y \right) V_0(y) \]
When \( V_0(y) = V_0 \),
\[ A_n = \frac{4V_0}{\pi \sinh \left( \frac{n\pi}{a} b \right)} \begin{cases} 0 & n \text{ even} \\ \frac{1}{n} & n \text{ odd} \end{cases} \]

19 Since the voltage should be finite as \( r \to 0 \) and \( r \to \infty \), the solutions are
\[ V_{\text{in}} = A_l r^l P_l(\cos \theta) \quad V_{\text{out}} = B_l r^{-(l+1)} P_l(\cos \theta) \]
At \( r = R \), we have, using the orthogonality of \( P_l \) functions,
\[ A_l = \frac{2l + 1}{2R^l} \int_0^\pi d\theta \sin \theta P_l(\cos \theta) k \cos 3\theta \]
To do these integrals, notice that
\[ \cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta = \frac{8}{5} P_3(\cos \theta) - \frac{3}{5} P_1(\cos \theta) \]
Therefore
\[ A_l = \frac{k(2l + 1)}{2R^l} \int_{-1}^{1} dx P_l(x) \left[ \frac{8}{5} P_3(x) - \frac{3}{5} P_1(x) \right] \]
This integral will be non-zero only for \( l = 1 \) and \( l = 3 \). We get
\[ A_1 = -\frac{3k}{5R} \quad A_3 = \frac{8k}{5R^3} \]
The continuity of the potential means \( A_l R^{2l+1} = B_l \). The continuity of the normal derivatives gives
\[ \sigma = -\epsilon_0 \left[ \frac{\partial V_{\text{out}}}{\partial r} - \frac{\partial V_{\text{in}}}{\partial r} \right] \bigg|_{r=R} = -\epsilon_0 \left[ B_1 (-2) R^{-3} P_1(\cos \theta) + B_3 (-4) R^{-5} P_3(\cos \theta) - A_1 (1) P_1(\cos \theta) - A_3 (3) P_3(\cos \theta) \right] \]
\[ = \frac{k\epsilon_0}{R} \left[ \frac{56}{5} P_3(\cos \theta) - \frac{9}{5} P_1(\cos \theta) \right] = \frac{k\epsilon_0}{R} (28 \cos^3 \theta - 15 \cos \theta) \]
Using the cylindrical basis functions,

\[ V = A_0 \ln s + B_0 + \sum_{m=1}^{\infty} (A_m s^m + B_m s^{-m})(C_m \cos m\phi + D_m \sin m\phi) \]

As \( s \to \infty \), \( V \to -E_0 x = -E_0 s \cos \phi \). Therefore, \( A_m = 0 \) for \( m \neq 1 \), and \( A_1 = -E_0 \). Also, \( B_0 = 0 \) and \( D_1 = 0 \). Since \( V = 0 \) at \( s = R \), \( B_m = 0 \) for \( m \neq 0 \) as well: we get \( A_1 R \cos \phi + B_1 R^{-1} \cos \phi = 0 \). Therefore \( B_1 = -A_1 R^2 = E_0 R^2 \) and

\[ V = -E_0 s \cos \phi \left( 1 - \frac{R^2}{s^2} \right) \]

The surface charge again comes through the normal derivatives:

\[ \sigma = -\epsilon_0 \left[ \frac{\partial V}{\partial s} \right]_{s=R} = 2\epsilon_0 E_0 \cos \phi \]

The quadrupole...

(a) With \( P_2(x) = \frac{1}{2}(3x^2 - 1) \) and \( \cos \alpha = \hat{r} \cdot \hat{r}' \), the quadrupole term is

\[ V_{\text{quad}} = \frac{1}{4\pi \epsilon_0} \frac{1}{r^3} \int d\nu' \frac{1}{2} \left[ 3(\hat{r} \cdot \hat{r}')^2 - 1 \right] r'^2 \rho(\nu') \]

Note, now, that

\[ (\hat{r} \cdot \hat{r}')^2 = \left( \sum_i \hat{r}_i \hat{r}'_i \right)^2 = \sum_{ij} \hat{r}_i \hat{r}'_i \hat{r}'_j \hat{r}_j = \sum_{ij} (\hat{r}_i \hat{r}_j)(\hat{r}'_i \hat{r}'_j) = \sum_{ij} (\hat{r}_i \hat{r}_j) \frac{(r'_i r'_j)}{r'^2} \]

\[ 1 = \hat{r} \cdot \hat{r} = \sum_i \hat{r}_i \hat{r}_i = \sum_{ij} \hat{r}_i \hat{r}_j \delta_{ij} \]

Putting these in the integral,

\[ V_{\text{quad}} = \frac{1}{4\pi \epsilon_0} \frac{1}{r^3} \sum_{ij} \hat{r}_i \hat{r}_j \int d\nu' \frac{1}{2} \left[ 3r'_i r'_j - r'^2 \delta_{ij} \right] \rho(\nu') \]
(b) In this case,
\[
\rho(r) = q \delta(z) \left[ \delta(x - \frac{a}{2}) \delta(y - \frac{a}{2}) + \delta(x + \frac{a}{2}) \delta(y + \frac{a}{2}) - \delta(x - \frac{a}{2}) \delta(y + \frac{a}{2}) - \delta(x + \frac{a}{2}) \delta(y - \frac{a}{2}) \right]
\]
This means that all \(z\)-components—all \(Q_{ij}\) where \(i = 3\) or \(j = 3\)—are zero. All diagonal components with \(i = j\) also turn out to be zero, because of the opposite sign charges. Also notice that the quadrupole moment is symmetric: \(Q_{ij} = Q_{ji}\). So we’re left with the non-zero components
\[
Q_{12} = Q_{21} = \int dv' \frac{3}{2} x'y' \rho(r') = \frac{3}{2} qa^2
\]

(c) When we shift the origin, \(r' \to r' - a\). Therefore, \(r_i' \to r_i' - a_i\) and
\[
r'_{ij} \to r_{ij}' - a_i r_{ij}' + a_ia_j \quad r'^2 \to \sum_i (r_i'^2 - a_i)^2 = \sum_i (r_i'^2 - 2a_i r_i' + a_i^2)
\]
Notice that all the terms added to the integral are either constants or linear in \(r_i'\). The constant terms give contributions proportional to
\[
\int dv' \rho(r') = Q = 0
\]
The linear terms give contributions proportional to
\[
\int dv' r_i' \rho(r') = p_i = 0
\]
Therefore the result is independent of the origin.

(d) Now, the octopole. We need \(P_3(x) = \frac{1}{2}(5x^3 - 3x)\). Then,
\[
(\hat{r} \cdot \hat{r}')^3 = \left( \sum_i \hat{r}_i \hat{r}_i' \right)^3 = \sum_{ijk} (\hat{r}_i \hat{r}_j \hat{r}_k)(\hat{r}_i' \hat{r}_j' \hat{r}_k') = \sum_{ijk} (\hat{r}_i \hat{r}_j \hat{r}_k) \frac{(r_i' r_j' r_k')}{r'^3}
\]
\[
\hat{r} \cdot \hat{r}' = \sum_i \hat{r}_i \hat{r}_i' = \sum_i \hat{r}_i' \left( \sum_{jk} \hat{r}_j \hat{r}_k \delta_{ij} \right) = \sum_{ijk} (\hat{r}_i \hat{r}_j \hat{r}_k) \frac{(r_i' \delta_{jk})}{r'}
\]
We then define

\[ V_{\text{oct}} = \frac{1}{4\pi \epsilon_0} \frac{1}{r^4} \sum_{ijk} \hat{r}_i \hat{r}_j \hat{r}_k O_{ijk} \]

where

\[ O_{ijk} = \int dv' \frac{1}{2} \left[ 5r'_i r'_j r'_k - 3r'^2 r'_\delta_{jk} \right] \rho (r') \]