Solutions to Assignment 6; Phys 185

1. (10 points) Find d_{cm}, the distance between the center of mass of the Earth-Moon system and the center of the Earth.

Answer: Set $x_e = 0$ and $x_m = d$.

$$d_{cm} = \frac{dM_m}{M_e + M_m} = 4.64 \times 10^6 \text{ m}$$

Note that $d_{cm} < R_e$.

2. (15 points) Find the moment of inertia I_o for the Earth-Moon system, where both the Earth and Moon revolve around their common center of mass. To do this accurately, you will need to use the "parallel axis theorem," which gives the moment of inertia of an object rotated about an axis that is parallel to an axis that passes through its center of mass:

$$I = I_{cm} + Mr^2$$

where I_{cm} is the moment of inertia about an axis through the center of mass, M is the mass, and r is the distance between the center of mass and the parallel axis of rotation.

Then, using I_o, calculate L_o, the orbital angular momentum due to the Earth and Moon revolving around their center of mass.

Answer: The Earth-Moon center of mass is the location of the parallel axis. Therefore,

$$I_e = \frac{2}{5} M_e R_e^2 + M_e d_{cm}^2$$

$$I_m = \frac{2}{5} M_m R_m^2 + M_e \left(d - d_{cm}\right)^2$$

$$I_o = I_e + I_m = 1.082 \times 10^{40} \text{ kg} \cdot \text{m}^2$$

Now, the orbital angular velocity of the earth-Moon system is $\omega_o = 2\pi/T_m$. Therefore

$$L_o = I_o \frac{2\pi}{T_m} = 2.882 \times 10^{34} \text{ kg} \cdot \text{m}^2/\text{s}$$

Don't forget that 1 day = $24 \times 60 \times 60$ seconds.

3. (5 points) Now calculate L_o again, making the simplifying assumption that the Earth is stationary and the Moon is a point mass that revolves around the Earth in a circle with radius d. Using the more accurate result from the previous question, calculate the percentage error you introduce into L_o if you make the assumption that the Moon revolves around the Earth.

Answer: With the simplifying assumption,

$$I_o \approx M_m d^2 = 1.085 \times 10^{40}\ \mathrm{kg \cdot m^2}$$

And therefore

$$L_o = I_o \frac{2\pi}{T_m} \approx 2.891 \times 10^{34}\ \mathrm{kg \cdot m^2/s}$$

The error introduced is:

$$\frac{2.891 - 2.882}{2.882} = 0.0031 = 0.31\%$$

Not a bad approximation.

4. (10 points) Calculate L_e, the angular momentum due to the Earth spinning around its own axis, and L_m, the angular momentum due to the Moon spinning around its own axis. Then, also using L_o from the previous questions, calculate L_T, the total angular momentum of the Earth-Moon system. Find what percentage of L_T is due to L_o, L_e, and L_m—and therefore determine which of these you can ignore.

Answer: The spin angular momenta are

$$L_e = \frac{2}{5} M_e R_e^2 \frac{2\pi}{T_e} = 7.058 \times 10^{33}\ \mathrm{kg \cdot m^2/s}$$

$$L_m = \frac{2}{5} M_m R_m^2 \frac{2\pi}{T_m} = 2.374 \times 10^{29}\ \mathrm{kg \cdot m^2/s}$$

Therefore

$$L_T = L_o + L_e + L_m = 3.558 \times 10^{34}\ \mathrm{kg \cdot m^2/s}$$

L_o is 80.3% of this total, while L_e is 19.7%. L_m, at $6.6 \times 10^{-4}\%$, is utterly negligible.

5. (15 points) The period T_m depends on d. Using the approximation that the Moon does uniform circular motion around the Earth, derive an equation expressing how T_m depends on d. Then plug the appropriate numbers into this equation to see if you do, in fact, get the correct result for T_m.

Answer: The total force on the Moon will be the gravitational attraction of the Earth. This must be providing the centripetal acceleration for uniform circular motion. Therefore, $\sum \vec{F} = M_m \vec{a}$ gives

$$G \frac{M_m M_e}{d^2} = M_m \omega_m^2 d$$

Cancelling the Moon's mass, using $\omega_m = 2\pi/T_m$ and solving for T_m, we get

$$T_m = 2\pi \sqrt{\frac{d^3}{G M_e}} = 2.367 \times 10^6\ \mathrm{s} = 27.4\ \mathrm{days}$$

Note that the error introduced is 0.37%, about what we would expect.

6. (5 points) Due to the tidal interactions that tidelock the Moon, the Earth's rate of rotation around its own axis has also been slowing down over the last few billion years. Due to conservation laws, this means d also changes. So in the past, when the Earth's day T_e' was shorter than T_e ($T_e' < T_e$), would the Earth-Moon distance d' have been larger or smaller than d today?

Answer: Use angular momentum conservation: L_T remains constant. If the Earth's rotation was faster in the past, this means that it's spin angular momentum must have been larger: $L_e' > L_e$. We've already established that L_m was negligible, that does not change. Therefore the orbital angular momentum must have been smaller in the past to conserve L_T: $L_o' < L_o$. From question 3, we see that this means that the Earth-Moon distance must have been smaller: $d' < d$.

7. (10 points) At a time in the past when T_e' was 20 hours, what would have been d', the distance between the Earth and Moon?

Answer: With a 20-hour day, we have $T_e' = \frac{5}{6}T_e$. Therefore, if we redo the calculation in question 4, we find that

$$L_e' = \frac{6}{5}L_e = 8.470 \times 10^{33} \text{ kg} \cdot \text{m}^2/\text{s}$$

Since $L_T' = L_T$, this means that

$$L_o' = L_T - L_e' = 2.741 \times 10^{34} \text{ kg} \cdot \text{m}^2/\text{s}$$

As in question 3, and using the equation from question 5,

$$L_o' = M_m d'^2 \frac{2\pi}{T_m'} = M_m \sqrt{GM_e d'} \quad \Rightarrow \quad d' = \frac{L_o'^2}{GM_m^2 M_e} = 3.477 \times 10^8 \text{ m}$$

This is closer than today, as it should be from question 6.

8. (15 points) Calculate the total energy of the Earth-Moon system today. For the gravitational potential energy, use $U = -Gm_1m_2/d_{12}$. Identify which of the energies are very small compared to the total energy, and are therefore negligible.

Answer: We need to account for the rotational kinetic energies associated with the Earth and Moon spinning around their own axes, the rotational kinetic energy due to the Moon orbiting the Earth, and the gravitational potential energy due to the attraction between the Earth and Moon.

$$K_e = \frac{1}{2}\left(\frac{2}{5}M_e R_e^2\right)\left(\frac{2\pi}{T_e}\right)^2 = 2.567 \times 10^{29} \text{ J}$$

$$K_m = \frac{1}{2}\left(\frac{2}{5}M_m R_m^2\right)\left(\frac{2\pi}{T_m}\right)^2 = 3.162 \times 10^{23} \text{ J}$$

$$K_o = \frac{1}{2}\left(M_m d^2\right)\left(\frac{2\pi}{T_m}\right)^2 = 3.822 \times 10^{28} \text{ J}$$

$$U = -G\frac{M_e M_m}{d} = -7.645 \times 10^{28} \text{ J}$$

Adding them all up,

$$E_T = K_e + K_m + K_o + U = 2.185 \times 10^{29} \text{ J}$$

Among all these energies, K_m is negligible.

9. (15 points) The tidal interactions that cause the spin of the Moon and Earth to slow down are like friction; they cause energy losses that are hard to account for. Calculate E_{loss}, the energy lost to tidal heating between the time when T'_e was 20 hours and today.

Answer: We now do the same energy calculations, but now with the past values d', T'_e, and, from question 5,

$$T'_m = 2\pi\sqrt{\frac{d^3}{GM_e}} = 2.040 \times 10^6 \text{ s}$$

Neglect, as established in question 8, K'_m.

$$K'_e = \frac{1}{2}\left(\frac{2}{5}M_e R_e^2\right)\left(\frac{2\pi}{T'_e}\right)^2 = 3.700 \times 10^{29} \text{ J}$$

$$K'_o = \frac{1}{2}\left(M_m d'^2\right)\left(\frac{2\pi}{T'_m}\right)^2 = 4.222 \times 10^{28} \text{ J}$$

$$U' = -G\frac{M_e M_m}{d'} = -8.443 \times 10^{28} \text{ J}$$

Adding them all up,

$$E'_T = K'_e + K'_o + U' = 3.278 \times 10^{29} \text{ J}$$

Energy conservation means that

$$E'_T = E_T + \Delta E_{th}$$

Therefore

$$E_{\text{loss}} = E'_T - E_T = 1.093 \times 10^{29} \text{ J}$$

Notice that $E_{\text{loss}} \geq 0$, as it has to be. If you get a negative number, you should check your work because there must be an error somewhere.