1. **(40 points)** It’s physically impossible to have a cold reservoir at absolute zero, but let’s see what would happen if such a thing were available.

You have a monatomic ideal gas that goes through the cycle $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 1$ shown in the diagram. No gas molecules are added or removed during the cycle.

Find everything $(W, \Delta U, Q)$ in terms of $p_0$ and $V_0$.

(a) The $1 \rightarrow 2$ part of the cycle takes place at constant temperature, so $T_1 = T_2$. The area under a constant temperature curve with temperature $T$ on the $p-V$ diagram, going from an initial $V_i$ to a final $V_f$, is

$$NkT \ln \left( \frac{V_f}{V_i} \right)$$

Find the work done by the gas for each step of this cycle: $W_{1\rightarrow2}$, $W_{2\rightarrow3}$, $W_{3\rightarrow4}$, $W_{4\rightarrow1}$.

**Answer:** Just take the negatives of the areas under the curves, and note that $pV = nRT$:

$$W_{1\rightarrow2} = -NkT \ln \left( \frac{2V_0}{V_0} \right) = -(2 \ln 2)p_0V_0$$

All the other areas are zero, so $W_{2\rightarrow3} = W_{3\rightarrow4} = W_{4\rightarrow1} = 0$.

(b) Find the change in thermal energy for each step: $\Delta U_{1\rightarrow2}$, $\Delta U_{2\rightarrow3}$, $\Delta U_{3\rightarrow4}$, $\Delta U_{4\rightarrow1}$.

**Answer:** For a monatomic ideal gas, $\Delta U = U_f - U_i = \frac{3}{2}nRT_f - \frac{3}{2}nRT_i = \frac{3}{2}(p_fV_f - p_iV_i)$. Therefore,

$$\Delta U_{1\rightarrow2} = \frac{3}{2}(2p_0V_0 - p_02V_0) = 0 \quad \Delta U_{2\rightarrow3} = \frac{3}{2}(0 - p_02V_0) = -3p_0V_0$$

$$\Delta U_{3\rightarrow4} = \frac{3}{2}(0 - 0) = 0 \quad \Delta U_{4\rightarrow1} = \frac{3}{2}(2p_0V_0 - 0) = 3p_0V_0$$
Notice that the $\Delta U$’s all sum to 0, which is as it should be for a cycle.

(c) Find the heat added to the gas for each step of this cycle: $Q_{1\rightarrow2}$, $Q_{2\rightarrow3}$, $Q_{3\rightarrow4}$, $Q_{4\rightarrow1}$.

**Answer:** Use $Q = \Delta U - W$:

$$Q_{1\rightarrow2} = \Delta U_{1\rightarrow2} - W_{1\rightarrow2} = (2 \ln 2)p_0V_0$$

$$Q_{2\rightarrow3} = \Delta U_{2\rightarrow3} - W_{2\rightarrow3} = -3p_0V_0$$

$$Q_{3\rightarrow4} = \Delta U_{3\rightarrow4} - W_{3\rightarrow4} = 0$$

$$Q_{4\rightarrow1} = \Delta U_{4\rightarrow1} - W_{4\rightarrow1} = 3p_0V_0$$

The sum of all $Q$’s is equal to the total $-W$, which is also as it should be.

(d) Find the total heat input to this gas in one cycle, $Q_{in}$. Also find the total heat removed from the gas, $Q_{out}$, and the total work done, $W$.

**Answer:** The total input is the sum of all positive $Q$ steps:

$$Q_{in} = Q_{1\rightarrow2} + Q_{4\rightarrow1} = (3 + 2(\ln 2))p_0V_0$$

The output is all the negative $Q$’s:

$$Q_{out} = -Q_{2\rightarrow3} = 3p_0V_0$$

The total work done by the gas is negative the total work done on the gas. The work done by the heat engine is the work done by the gas.

$$W = (2 \ln 2)p_0V_0$$

Notice that $Q_{in} = Q_{out} + W$, as it should be for a heat engine.

(e) What is the efficiency of this heat engine? (Your result should be a number.)

**Answer:**

$$e = \frac{W}{Q_{in}} = \frac{2(\ln 2)}{3 + 2(\ln 2)} = 0.32$$

2. **(40 points)** If you look up how convection works, you will find $dQ/dt = hA\Delta T$, where $A$ is the surface area of an object, and $h$ is a convection coefficient that depends on the material and its geometric shape. You know how conduction and radiation works.
(a) You have two cubes made of identical materials, in identical environments, at identical starting temperatures. Cube 1 has a side of length \(a\), cube 2 has \(2a\). Find the ratio of the rates at which each cube cools:

\[
\left(\frac{\Delta T_1}{\Delta t}\right) : \left(\frac{\Delta T_2}{\Delta t}\right)
\]

Note: \(\Delta T\) refers to the temperature difference with the environment. \(\Delta T_1\) and \(\Delta T_2\) are different—they refer to the change in temperature over time of cubes 1 and 2. Also, you’ll need this: for the small temperature ranges in question, \(dQ/dt = Q/\Delta t\).

Hint: Your final result should be a number, with no symbols. Cancel things!

Answer: Add up the heat loss rates for cube 1:

\[
dQ_1 \over dt = \frac{Q_1}{\Delta t} = \frac{kA_1 \Delta T}{L} + hA_1 \Delta T + e\sigma A_1 \Delta T^4 = \left[\frac{k\Delta T}{L} + h\Delta T + e\sigma \Delta T^4\right] A_1
\]

Notice that the factor in brackets depends on the material of the cube and its surrounding environment—not the size of the cube! Calling the bracketed factor \(\alpha\), we then have

\[
\frac{Q_1}{\Delta t} = \alpha A_1 \quad \text{and} \quad \frac{Q_2}{\Delta t} = \alpha A_2
\]

For the temperature change, we use

\[
Q_1 = m_1 c \Delta T_1 = \rho V_1 c \Delta T_1 = [\rho c] V_1 \Delta T_1
\]

where \(\rho\) is the density of the material, and \(V_1\) is the volume of cube 1. Again, the bracketed factor depends only on the material, and therefore is the same for both cubes. Call it \(\beta\). So

\[
Q_1 = \beta V_1 \Delta T_1 \quad \text{and} \quad Q_2 = \beta V_2 \Delta T_2
\]

Putting the heat loss and the temperature equations together, we get

\[
\frac{\beta V_1 \Delta T_1}{\Delta t} = \alpha A_1
\]

With some rearranging,

\[
\frac{\Delta T_1}{\Delta t} = \frac{\alpha A_1}{\beta V_1} \quad \text{and} \quad \frac{\Delta T_2}{\Delta t} = \frac{\alpha A_2}{\beta V_2}
\]

Finally, when we take the ratio, the constants \(\alpha\) and \(\beta\) cancel out, and we’re just left with the ratios of surface-area-to-volume ratios:

\[
\frac{\Delta T_1}{\Delta t} : \frac{\Delta T_2}{\Delta t} = \frac{A_1}{\beta V_1} : \frac{A_2}{\beta V_2} = \frac{6a^2}{6(2a)^2} = \frac{\frac{6a^2}{(2a)^2}}{2} = 2
\]

The small cube will cool down twice as fast.
(b) Use this to predict whether in cold climates, small or large animals will have proportionally thicker coats, and area-reducing adaptations such as smaller external ears. Explain.

**Answer:** The result of (a) means that smaller animals cool down faster. So they have higher adaptive pressure on them to reduce their heat loss—proportionally thicker coats, and smaller external ears and so forth.

3. **(20 points)** Take a small bubble of air at a depth $d$ below the ocean surface. There are $N$ molecules of air in the bubble, and air is approximated very well as an ideal gas. Let’s assume that the bubble is small enough that we can assume a single depth and a single pressure value accurately characterizes the bubble. Let’s also assume that the ocean has a constant temperature $T$ at any depth, and that the air is always in thermal equilibrium with the ocean. Use $p_{atm}$ to represent atmospheric pressure and $\rho_w$ to represent the density of water.

(a) Write down an equation for $V$, the volume of the bubble.

**Answer:** Use $pV = NkT$, and the fact that the pressure within the bubble will be equal to the water pressure at its depth, $p = p_{atm} + \rho_w gd$:

$$V = \frac{NkT}{p_{atm} + \rho_w gd}$$

(b) Now write down an equation for the buoyancy force $F_B$ on the bubble.

**Answer:** The buoyancy force is equal to the weight of water of an equal volume:

$$F_B = \rho_w V g = \frac{NkT \rho_w g}{p_{atm} + \rho_w gd}$$

(c) Make a rough sketch of the buoyancy force versus depth. Make sure the sketch is clear about whether $F_B$ almost at the surface ($d = 0$) is zero, infinite, or a finite value.

**Answer:** The curve for $F_B$ should begin with a finite value for $d = 0$, and monotonically decrease without ever becoming zero.