Solutions to Assignment 4; PHYS 185

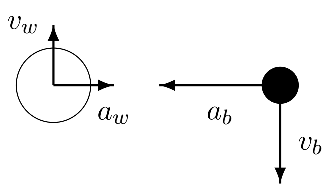
1. (30 points) You have a bucket tethered to a rope, and you swing the rope such that the bucket rotates in a vertical circle with radius r. There is a small ball with mass m in the bucket. What is the minimum v at the top of the circle that you must have in order for the ball not to fall out of the bucket during rotation?

Answer: The forces on the ball are its weight and the normal force from the bottom of the bucket. At the top of the circle, both the weight and the normal force point down toward the center of the circle, so at that instant, *uniform* circular motion applies. ($\vec{n}$ cannot point up; it has to be out of the surface, not into the surface.) The ball will be barely touching the bottom of the bucket if $\vec{n} = 0$, and if an upward pointing $\vec{n}$ is necessary for the ball to keep in a circle, that means the ball will fall out. Therefore, we look for uniform circular motion at the top with $\vec{n} = 0$ to find the minimum v.

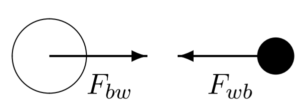
$$\sum F_y = w_y + n_y = -mg - n = -mg = -m\frac{v^2}{r} \quad \Rightarrow \quad v = \sqrt{rg}$$

2. (30 points) The diagrams below show a binary star system. The white star is more massive than the one shown as a black circle, but not hugely more massive. The stars revolve in circular orbits around their common center of mass. On diagram (a), draw and label the velocity and acceleration vectors for each star. On (b), show the forces on each star. Assume that the stars are in deep space and we can ignore the effects of the rest of the universe on either star. Draw the sizes of your arrows such that I can tell whether v, a, and F for each star is larger, smaller, or equal to the other.

(a)



(b)



Answer: Let's start with the forces. These are action-reaction pairs, therefore they will be equal and opposite to each other—the stars gravitationally attract one another.

Gravity is the only force on each star. Therefore for each of them, $\vec{a} = \vec{F}/m$. The forces have the same magnitude, but the white star is more massive; therefore its acceleration a_w will be smaller in magnitude.

The orbits are circular, therefore $a = v^2/r$ and $F = mv^2/r$, with r the distance to the center of mass. We therefore have

$$v = \sqrt{\frac{rF}{m}}$$

for each star. F is the same for both stars. But since the center of mass is closer to the more massive star, r_w is smaller for the white star, and m_w is larger. That means that v_w is smaller for the white star.

Alternatively: the period T of the circular motions of each star are the same. Since $v = 2\pi r/T$, that means the star closer to the center of mass must be going slower.

3. (40 points) Astronomers observe a planet that has a small moon with a circular orbit. Call the planet's mass M and the moon's mass m; from the orbit, astronomers can also tell that $m \ll M$. Astronomers can also observe r, the distance between the planet and the moon, and T, the period of the moon's orbit around the planet. They also know that gravity is the only significant force between the moon and the planet.

(a) Given all the above information, can the astronomers determine M? If so, find an equation for M.

Answer: Since $M \gg m$, we can take the moon to be in a circular orbit centered on the planet, with radius r. Then,

$$\sum \vec{F} = m\vec{a} \qquad \Rightarrow \qquad G\frac{Mm}{r^2} = m\frac{v^2}{r} = m\frac{(2\pi/T)^2}{r} = \frac{4\pi^2 mr}{T^2}$$

Canceling out the m's and rearranging,

$$M = \frac{4\pi^2 r^3}{GT^2}$$

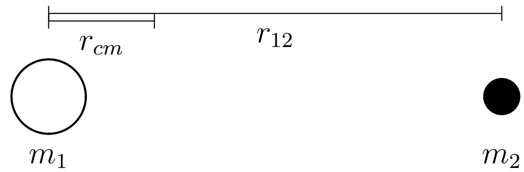
So the mass of the planet can be inferred from astronomical observations.

(b) Given all the above information, can the astronomers determine m? If so, find an equation for m.

Answer: In the above calculation, m cancels out. So without some other source of information, it won't be possible to figure out the mass of the moon.

Extra Problems (not graded)

4. (0 points) The diagram below shows a binary star system. The white star has mass m_1, the black star mass m_2. The distance between the centers of stars is r_{12}. The only force on each star is the gravity from the other star. You look up the center of mass for two bodies and find that the center of mass is located at a distance $r_{cm} = \frac{m_2}{m_1+m_2} r_{12}$ from the center of the white star.



(a) Find equations for a_1, a_2, v_1, v_2—the magnitudes of the accelerations of each star and the speeds of each star, in terms of m_1, m_2, r_{12}, and appropriate physical constants.

Answer: The force of attraction between the stars is $F_G = G\,m_1 m_2/r_{12}^2$. The accelerations are just that force divided by each mass:

$$a_1 = G\,\frac{m_2}{r_{12}^2} \qquad a_2 = G\,\frac{m_1}{r_{12}^2}$$

For circular motion, $a = v^2/r$, and so $v = \sqrt{ar}$. Therefore

$$v_1 = \sqrt{G\,\frac{m_2}{r_{12}^2}\,r_{cm}} = \sqrt{G\,\frac{m_2}{r_{12}^2}\,\frac{m_2}{m_1+m_2}\,r_{12}} = \sqrt{\frac{G\,m_2^2}{r_{12}(m_1+m_2)}}$$

$$v_2 = \sqrt{G\,\frac{m_1}{r_{12}^2}\,(r_{12}-r_{cm})} = \sqrt{G\,\frac{m_1}{r_{12}^2}\,\frac{m_1}{m_1+m_2}\,r_{12}} = \sqrt{\frac{G\,m_1^2}{r_{12}(m_1+m_2)}}$$

(b) To confirm your results, show that the period of the stars—the time it takes for each to make one full circle—is the same.

Answer: The periods being the same means that $2\pi r/v$ for each orbit is the same. Therefore,

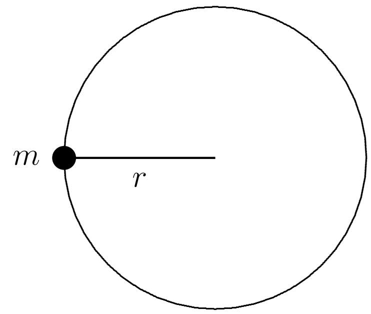
$$\frac{2\pi r_{cm}}{v_1} = \frac{2\pi(r_{12}-r_{cm})}{v_2} \quad\Rightarrow\quad \frac{m_2}{m_1+m_2}\,\frac{1}{v_1} = \frac{m_1}{m_1+m_2}\,\frac{1}{v_2} \quad\Rightarrow\quad \frac{v_1}{v_2} = \frac{m_2}{m_1}$$

If you take the ratio of the answers from (a), we get

$$\frac{v_1}{v_2} = \sqrt{\frac{m_2^2}{m_1^2}} = \frac{m_2}{m_1}$$

So it checks out.

5. (0 points) You have a mass attached to a string, and you set the mass in motion such that it goes in a vertical circle (in the xy-plane, so it keeps changing height).



(a) Explain why the circular motion of the mass cannot be *uniform* circular motion.

Answer: There are two forces on the mass: weight and tension. The weight always points downward, and the tension always points radially, toward the center of the circle. For *uniform* circular motion, the total force must be constant in magnitude and must point toward the center of the circle. In this case, you can only get a radial total force at the very top and very bottom of the circle. At all other points, the weight will not be radial, and therefore the total force cannot have a radial direction.

(b) At what point on the circle is the tension in the string the largest? Why?

Answer: At the very bottom. There, the weight is radial and it opposes the tension. To keep the mass moving in a circle, the tension force must cancel out the weight and have some left over to provide the total force directed toward the center.

(c) Find an equation for $T_{\max}$, the largest tension in the string during the rotation. $T_{\max}$ might depend on m (the mass of your mass), r (the radius of the circle), g (the acceleration due to gravity), v (the speed of your mass at the point where the tension was largest), and k (the spring constant).

Answer: At the very bottom, the forces have $w_y = -mg$ and $T_y = T$. Since the total force is radial, at that point, for that instant, the ball will be doing uniform circular motion, leading to $a_y = mv^2/r$. Summing the forces:

$$\sum F_y = ma_y \quad \Rightarrow \quad T - mg = m\frac{v^2}{r} \quad \Rightarrow \quad T = m\left(g + \frac{v^2}{r}\right)$$

We've established that the largest tension is at the bottom, so

$$T_{\text{max}} = m\left(g + \frac{v^2}{r}\right)$$