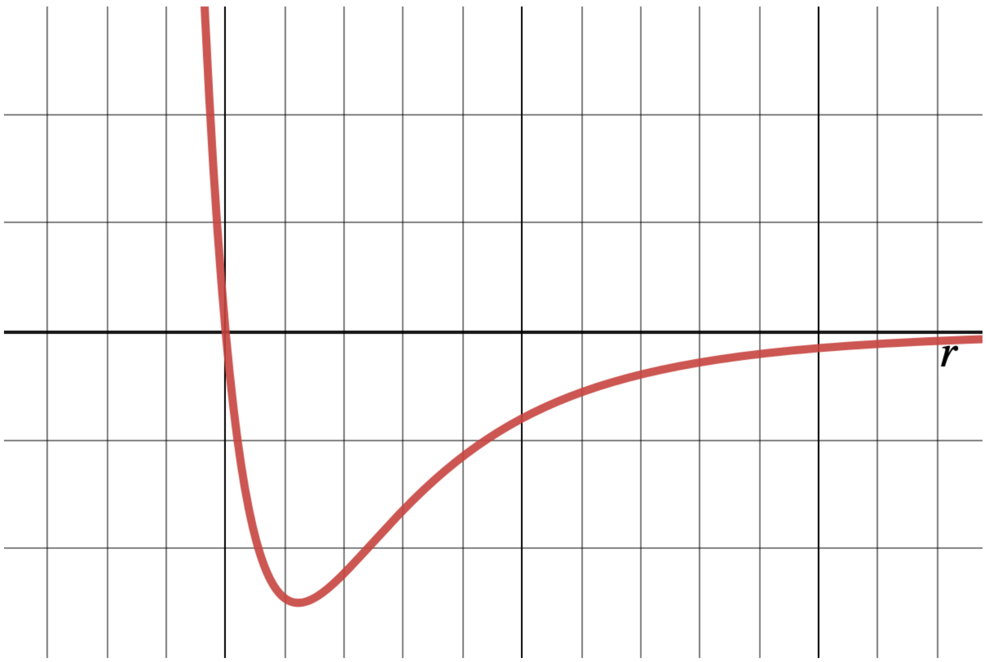
Solutions to Assignment 8; PHYS 185

1. (30 points) The following graph gives the potential energy $U(r)$ due to the interaction between two particles, where r is the distance between the particles. Notice that U approaches 0 when r is very large, and that U becomes very large when r approaches 0.

The total energy of a pair of particles is $E_T = U + K$, where K is the kinetic energy due to the relative motion of the particles. There are no forces other than the interaction described by $U(r)$.

The two particles are said to be bound to each other if it is impossible for r to be larger than an upper limit, r_{max}. If there is no upper limit—if r can be arbitrarily large—the particles are unbound.



(a) Say $E_T > 0$. Are the particles bound or unbound? Provide an argument. (A visual argument where you draw on the graph above will be fine, if you prefer.)

Answer: Unbound. Use energy conservation, since there are no dissipative forces that produce any E_{loss}. There are no accounting difficulties.

As r gets large, the most $U(r)$ can become is 0. Therefore, r can get arbitrarily large and $E_T > 0$ can be satisfied with a $K > 0$, which is fine.

(b) Say $E_T < 0$. Are the particles bound or unbound? Again, provide an argument.

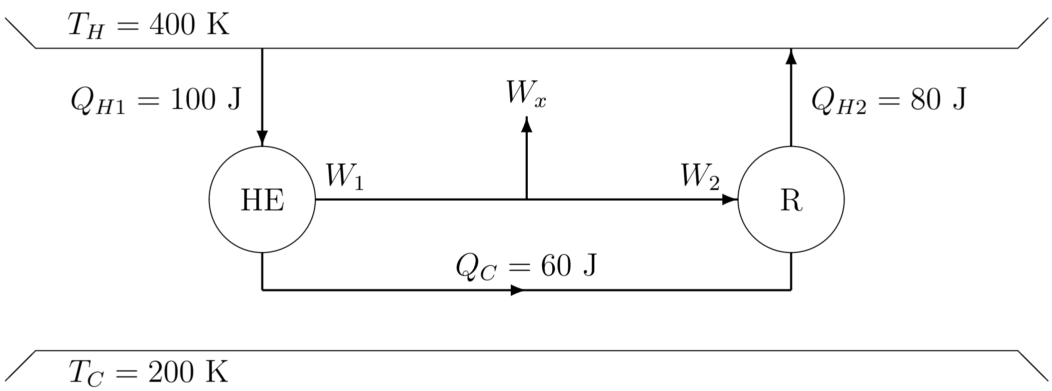
Answer: Bound.

Since it is only possible that $K = \frac{1}{2}mv^2 \geq 0$, at large r, the negative E_T has to come from $U < 0$. That is the case only up to a finite value of r_{max}. If r gets too large, U starts to get too close to 0.

(c) Say you put lots and lots of particles together, where the interaction of each pair of particles is described as above, and where the temperature is such that the *average* total energy $\bar{E}_T > 0$. Will this population of particles be in a gaseous state, or will it be in a more condensed state (liquid or solid)? Explain.

Answer: On average, the particles will be unbound. There is too much thermal energy in the system for the particles to consistently stick together; they will wander apart to arbitrarily large distances. That is the description of a gas.

2. (30 points) An inventor asks you to support his "free energy" device. He has this brilliant idea of using the exhaust heat and work from a heat engine as input to a refrigerator. He says that in each cycle, his heat engine takes in 100 J from a 400 K heat reservoir, and exhausts 60 J. But instead of just dumping that 60 J into a 200 K cold reservoir, he feeds that directly into a refrigerator. He also uses the work output by the heat engine to run the refrigerator, but since the refrigerator only returns 80 J per cycle to the hot reservoir, there is an excess work W_x that is left over for you to use as you like. You will never pay an electric bill again!



(a) Should you support this invention? Show your calculations.

Answer: There are a number of ways to answer this. For example, you could calculate the entropies, ending with a total

$$\Delta S_{\mathsf{total}} = -\frac{Q_{H1}}{T_H} + \frac{Q_{H2}}{T_H} = -\frac{100}{400} + \frac{80}{400} = -\frac{20}{400} = -0.05\,\mathrm{J/K} < 0$$

A negative change in total entropy violates the Second Law of Thermodynamics.

Alternatively, you can use energy conservation, finding $W_1 = Q_{H1} - Q_C = 40$ J; $W_2 = Q_{H2} - Q_C = 20$ J; $W_x = W_1 - W_2 = 20$ J.

The maximum efficiency for a heat engine operating between these reservoirs is $e_{\max} = 1 - \frac{T_C}{T_H} = 0.5$. The actual efficiency is $e = \frac{W_1}{Q_{H1}} = 0.4 < e_{\max}$, so this part of the device is fine.

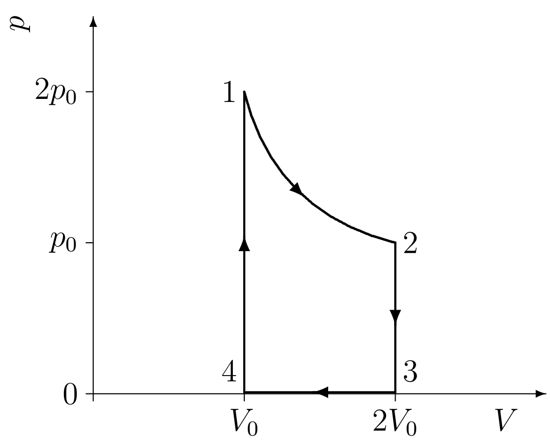
The maximum COP for a refrigerator operating between these reservoirs is $\text{COP}_{\max} = \frac{T_C}{T_H - T_C} = 1$. The actual COP $= \frac{Q_C}{W_2} = 3 > \text{COP}_{\max}$. So the refrigerator component of this device is impossible.

(b) If in (a), you found that this invention was workable, are there any values of Q_C and T_C for which the invention wouldn't work? If your answer to (a) was "no," are there any values of Q_C and T_C for which the invention would work? Explain.

Answer: No matter what Q_C and T_C are, this device extracts a net heat of $Q_{H1} - Q_{H2} = 20$ J from the hot reservoir, converting it into $W_x = 20$ J of work without any exhaust heat. Such a free conversion of heat into work is impossible—it violates the Second Law of Thermodynamics, as we saw in (a).

3. (40 points) It's physically impossible to have a cold reservoir at absolute zero, but let's see what would happen if such a thing were available.

You have a monatomic ideal gas that goes through the cycle $1 \to 2 \to 3 \to 4 \to 1$ shown in the diagram. No gas molecules are added or removed during the cycle.



Find everything (W, ΔU, Q) in terms of p_0 and V_0.

(a) The $1 \to 2$ part of the cycle takes place at *constant temperature*, so $T_1 = T_2$. The area under a constant temperature curve with temperature T on the p-V diagram, going from from an initial V_i to a final V_f, is

$$NkT \ln\left(\frac{V_f}{V_i}\right)$$

Find the work done by the gas for each step of this cycle: $W_{1\to2}$, $W_{2\to3}$, $W_{3\to4}$, $W_{4\to1}$.

Answer: Just take the negatives of the areas under the curves, and note that $pV = nRT$:

$$W_{1\to2} = -NkT\ln\left(\frac{2V_0}{V_0}\right) = -(2\ln 2)p_0V_0$$

All the other areas are zero, so $W_{2\to3} = W_{3\to4} = W_{4\to1} = 0$.

(b) Find the change in thermal energy for each step: $\Delta U_{1\to2}$, $\Delta U_{2\to3}$, $\Delta U_{3\to4}$, $\Delta U_{4\to1}$.

Answer: For a monatomic ideal gas, $\Delta U = U_f - U_i = \frac{3}{2}nRT_f - \frac{3}{2}nRT_i = \frac{3}{2}(p_fV_f - p_iV_i$. Therefore,

$$\Delta U_{1\to2} = \frac{3}{2}(2p_0V_0 - p_02V_0) = 0 \qquad \Delta U_{2\to3} = \frac{3}{2}(0 - p_02V_0) = -3p_0V_0$$

$$\Delta U_{3\to4} = \frac{3}{2}(0 - 0) = 0 \qquad \Delta U_{4\to1} = \frac{3}{2}(2p_0V_0 - 0) = 3p_0V_0$$

Notice that the ΔU's all sum to 0, which is as it should be a for a cycle.

(c) Find the heat added to the gas for each step of this cycle: $Q_{1\to2}$, $Q_{2\to3}$, $Q_{3\to4}$, $Q_{4\to1}$.

Answer: Use $Q = \Delta U - W$:

$$Q_{1\to2} = \Delta U_{1\to2} - W_{1\to2} = (2\ln 2)p_0V_0 \qquad Q_{2\to3} = \Delta U_{2\to3} - W_{2\to3} = -3p_0V_0$$

$$Q_{3\to4} = \Delta U_{3\to4} - W_{3\to4} = 0 \qquad Q_{4\to1} = \Delta U_{4\to1} - W_{4\to1} = 3p_0V_0$$

The sum of all Q's is equal to the total $-W$, which is also as it should be.

(d) Find the total heat input to this gas in one cycle, Q_{in}. Also find the total heat removed from the gas, Q_{out}, and the total work done, W.

Answer: The total input is the sum of all positive Q steps:

$$Q_{\text{in}} = Q_{1\to2} + Q_{4\to1} = [3 + 2(\ln 2)]p_0V_0$$

The output is all the negative Q's:

$$Q_{\text{out}} = -Q_{2\to3} = 3p_0V_0$$

The total work done *by* the gas is negative the total work done *on* the gas. The work done by the heat engine is the work done *by* the gas..

$$W = (2\ln 2)p_0V_0$$

Notice that $Q_{\text{in}} = Q_{\text{out}} + W$, as it should be for a heat engine.

(e) What is the efficiency of this heat engine? (Your result should be a number.)

Answer:

$$e = \frac{W}{Q_{\text{in}}} = \frac{2(\ln 2)}{3 + 2(\ln 2)} = 0.32$$

Extra Problems (not graded)

4. (0 points) You have a heat engine that extracts heat at a rate of 50 kW from a heat reservoir at 200°C, and exhausts 40 kW to a reservoir at 50°C. You then decide that you can use the heat going to waste, so you attach a second heat engine to the exhaust of the first: it takes the 40 kW as input, and finally dumps its own exhaust into a reservoir at room temperature, 20°C.

(a) If the second heat engine is reversible—it produces the least possible amount of exhaust heat—what is the power it produces? What is the rate at which it exhausts heat?

Answer: With the least amount of exhaust, you have maximum efficiency,

$$e = 1 - \frac{20 + 273}{50 + 273} = 0.093$$

With this, the power produced is $e(40) = 3.7$ kW, and the exhaust heat is $40 - 3.7 = 36.3$ kW.

(b) You can think of the combination of the first and second heat engines as a single combined heat engine operating between 200°C and 20°C, producing a total power which is the sum of their outputs. What is the efficiency of this combined heat engine?

Answer: The combined efficiency is the total power divided by the input rate of heat. The first stage produces $50 - 40 = 10$ kW, so

$$e_c = \frac{10 + 3.7}{50} = 0.274$$

5. (0 points) Give an example of an irreversible process—but come up with an example I didn't use in class. In other words, no shattered objects re-forming themselves spontaneously, no billiard balls etc. How can you tell that your example is irreversible?

Answer: Just about anything macroscopic will serve as an example of irreversibility. You tell by imagining a movie of the process run backwards in time and seeing if you can tell if something is strange.