Solutions to Assignment 9; PHYS 185

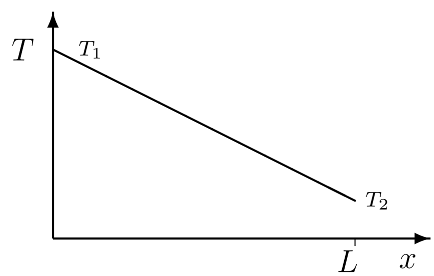
1. (30 points) You have a slab of material through which heat is being conducted at a rate of dQ/dt. The thickness of the material is L, the temperature on one side is T_1 and on the other side it's T_2.

(a) What is the temperature in the middle of the slab, a distance $L/2$ from either side? *Hint:* The same heat is going through at the same rate through both halves of the slab.

Answer: Since both halves of the slab are identical, and they have the same heat flow through them, they must have the same temperature difference across them. Therefore the temperature at the halfway point must be halfway between the temperatures of either end. If we express that more mathematically, we have, with T_m the middle temperature,

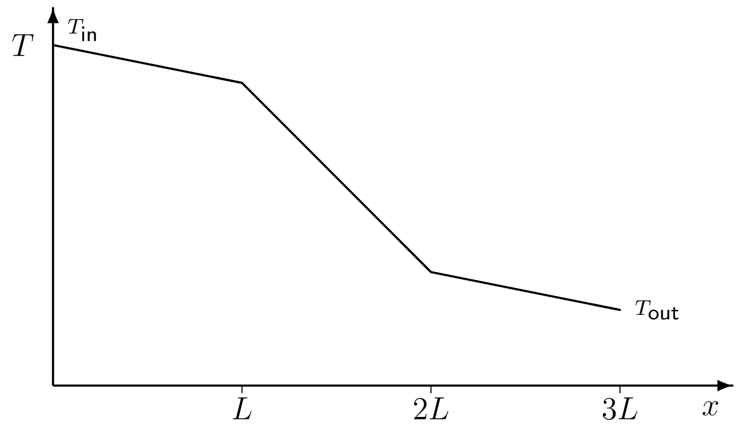
$$k\frac{A}{L/2}(T_1 - T_m) = k\frac{A}{L/2}(T_m - T_2) \quad \Rightarrow \quad T_1 - T_m = T_m - T_2 \quad \Rightarrow \quad T_m = \frac{T_1 + T_2}{2}$$

(b) Make a graph of $T(x)$, the temperature within the slab, with $x = 0$ one side of the slab and $x = L$ the other side. Give your reasoning for the shape you drew.



Answer: The argument in (a) applies if you take each half and halve them in turn, making quarters. And so forth. So the temperature profile has to be a straight line connecting the temperatures at the ends.

(c) You have a double-glazed window between the inside of a room at T_{in} and the outside at T_{out}. The thickness of the the two glass panes and the air trapped is all equal, L. The thermal conductivity of air is less than glass: $k_a < k_g$. Make a graph of $T(x)$, the temperature within the window, with $x = 0$ one side of the slab and $x = 3L$ the other side. Give your reasoning for what you drew.



Answer: The same heat is going through all three layers. The difference now is that there are different thermal conductivities in play. Call the temperatures at the glass-air interfaces T_{m1} and T_{m2}, we have

$$k_g \frac{A}{L}(T_{\text{in}} - T_{m1}) = k_a \frac{A}{L}(T_{m1} - T_{m2}) = k_g \frac{A}{L}(T_{m2} - T_{\text{out}})$$

With the A's and L's canceling out, this means that the temperature difference across the glass panes is smaller than the temperature difference across the air. We've already established that within the same material, the temperature changes in a linear fashion.

2. (40 points) If you look up how convection works, you will find $dQ/dt = hA\Delta T$, where A is the surface area of an object, and h is a convection coefficient that depends on the material and its geometric shape. You know how conduction and radiation works.

(a) You have two cubes made of identical materials, in identical environments, at identical starting temperatures. Cube 1 has a side of length a, cube 2 has $2a$. Find the ratio of the rates at which each cube cools:

$$\frac{\left(\frac{\Delta T_1}{\Delta t}\right)}{\left(\frac{\Delta T_2}{\Delta t}\right)}$$

Note: ΔT refers to the temperature difference with the environment. ΔT_1 and ΔT_2 are *different*—they refer to the change in temperature over time of cubes 1 and 2. Also, you'll need this: for the small temperature ranges in question, $dQ/dt = Q/\Delta t$.

Hint: Your final result should be a number, with no symbols. Cancel things!

Answer: Add up the heat loss rates for cube 1:

$$\frac{dQ_1}{dt} = \frac{Q_1}{\Delta t} = \frac{kA_1\Delta T}{L} + hA_1\Delta T + e\sigma A_1 \Delta T^4 = \left[\frac{k\Delta T}{L} + h\Delta T + e\sigma \Delta T^4\right] A_1$$

Notice that the factor in brackets depends on the material of the cube and its surrounding environment—not the size of the cube! Calling the bracketed factor α, we then have

$$\frac{Q_1}{\Delta t} = \alpha A_1 \qquad \text{and} \qquad \frac{Q_2}{\Delta t} = \alpha A_2$$

For the temperature change, we use

$$Q_1 = m_1 c \Delta T_1 = \rho V_1 c \Delta T_1 = [\rho c] V_1 \Delta T_1$$

where ρ is the density of the material, and V_1 is the volume of cube 1. Again, the bracketed factor depends only on the material, and therefore is the same for both cubes. Call it β. So

$$Q_1 = \beta V_1 \Delta T_1 \qquad \text{and} \qquad Q_2 = \beta V_2 \Delta T_2$$

Putting the heat loss and the temperature equations together, we get

$$\frac{\beta V_1 \Delta T_1}{\Delta t} = \alpha A_1$$

With some rearranging,

$$\frac{\Delta T_1}{\Delta t} = \frac{\alpha}{\beta}\frac{A_1}{V_1} \qquad \text{and} \qquad \frac{\Delta T_2}{\Delta t} = \frac{\alpha}{\beta}\frac{A_2}{V_2}$$

Finally, when we take the ratio, the constants α and β cancel out, and we're just left with the ratios of surface-area-to-volume ratios:

$$\frac{\left(\frac{\Delta T_1}{\Delta t}\right)}{\left(\frac{\Delta T_2}{\Delta t}\right)} = \frac{\frac{\alpha}{\beta}\frac{A_1}{V_1}}{\frac{\alpha}{\beta}\frac{A_2}{V_2}} = \frac{\frac{A_1}{V_1}}{\frac{A_2}{V_2}} = \frac{\frac{6a^2}{a^3}}{\frac{6(2a)^2}{(2a)^3}} = 2$$

The small cube will cool down twice as fast.

(b) Use this to predict whether in cold climates, small or large animals will have proportionally thicker coats, and area-reducing adaptations such as smaller external ears. Explain.

Answer: The result of (a) means that smaller animals cool down faster. So they have higher adaptive pressure on them to reduce their heat loss—proportionally thicker coats, and smaller external ears and so forth.

3. (30 points) Take a small bubble of air at a depth d below the ocean surface. There are N molecules of air in the bubble, and air is approximated very well as an ideal gas. Let's assume that the bubble is small enough that we can assume a single depth and a single pressure value accurately characterizes the bubble. Let's also assume that the ocean has a constant temperature T at any depth, and that the air is always in thermal equilibrium with the ocean. Use p_{atm} to represent atmospheric pressure and ρ_w to represent the density of water.

(a) Write down an equation for V, the volume of the bubble.

Answer: Use $pV = NkT$, and the fact that the pressure within the bubble will be equal to the water pressure at its depth, $p = p_{atm} + \rho_w g d$:

$$V = \frac{NkT}{p_{atm} + \rho_w g d}$$

(b) Now write down an equation for the buoyancy force F_B on the bubble.

Answer: The buoyancy force is equal to the weight of water of an equal volume:

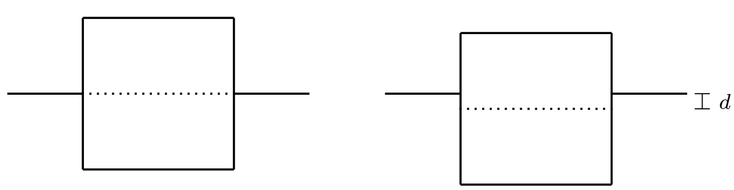
$$F_B = \rho_w V g = \frac{NkT\rho_w g}{p_{atm} + \rho_w g d}$$

(c) Make a rough sketch of the buoyancy force versus depth. Make sure the sketch is clear about whether F_B almost at the surface ($d = 0$) is zero, infinite, or a finite value.

Answer: The curve for F_B should begin with a finite value for $d = 0$, and monotonically decrease without ever becoming zero.

Extra Problems (not graded)

4. (0 points) You have a cube with sides a floating in water, with density ρ_w. You mark where the water line is on the cube. Then, you push down the cube by an extra depth d, and let go of the cube. Assume that $d \ll a$ and that you're able to uniformly push the cube down and so forth—in other words, assume that drag forces and any other complications are negligible.



(a) Find what the total force on the cube is at the instant you let it go at depth d.

Answer: The only forces are the weight of the cube and the buoyancy force. We then use $\sum F_y = F_B - w = ma_y$. When the cube was floating, $\sum F_y = 0$, and therefore F_B canceled out w. But when you push the cube down an extra depth d, it picks up extra buoyancy without changing its weight. The extra buoyancy is the weight of water with volume equal to the extra submerged volume:

$$\sum F_y = F_B - w = \rho_w(a^2 d)g = ma_y$$

The result is a total force that points directly upward.

(b) When $d = 0$, what should the total force be? Check if your answer to (a) behaves accordingly.

Answer: When $d = 0$, F_B cancels out w, making the total force 0. The answer to (a), $\rho_w a^2 dg$, becomes 0 when $d = 0$, which is therefore as it should be.

(c) Your total force depends on d in a way precisely analogous to how another force we have examined this semester depends on a displacement from equilibrium. Identify this force. Then, using this analogy, qualitatively describe the motion your cube will exhibit once you let it go.

Answer: The other force that behaves the same way is the spring force. The spring force is proportional to the extension and compression, and is a restoring force—it is opposite to the displacement, toward equilibrium.

You know from your experience with springs that if you extend a spring with a mass attached and let it go, it will oscillate up and down. The floating cube will bob up and down in the water in exactly the same way.

(d) Write down an equation giving the period of the oscillations of the cube.

Answer: With a spring, we had $-kx = ma_x$. With the floating cube, d is a *downward* displacement, so $d = -y$. We can therefore write the equation describing the cube's motion as $-(\rho_w a^2 g)y = ma_y$. This is exactly the same as the spring equation, with the constant $\rho_w a^2 g$ playing the role of the spring constant. Since for a spring, $T = 2\pi\sqrt{m/k}$, the period of the floating cube must be

$$T = 2\pi\sqrt{\frac{m_c}{\rho_w a^2 g}} = 2\pi\sqrt{\frac{\rho_c a}{\rho_w g}}$$

where m_c is the mass of the cube, or ρ_c is the density of the cube.

5. (0 points) If a space traveler were to step out into the vacuum of empty space, far away from all stars and other sorts of heat, she would be in an environment close to absolute zero: the 3 K of the cosmic microwave background radiation, in fact. To calculate if she would freeze to death, you gather the following information. The thermal conductivity of her clothes: 0.05 W/m·K. Thickness of her clothes: 0.010 m. Her surface area: 1.8 m^2. The emissivity of her clothes surface: 0.5. Stefan-Boltzmann constant: 5.67×10^{-8} W/m^2·K^4. The surface temperature of her clothes: 35°C. The specific heat of mammalian tissue: 3400 J/kg·K. Her mass: 56.0 kg. Rate of thermal energy generation by a human body at rest: 100 W. (Not all these values are necessarily useful!)

(a) Calculate the rate the astronaut absorbs heat from the microwave background.

Answer: With $T_\mu = 3$ K,

$$\frac{dQ}{dt} = e\sigma A T_\mu^4 = 4.13 \times 10^{-6} \text{ W}$$

This is miniscule, as you might expect.

(b) Calculate her *net* rate of heat loss.

Answer: In outer space, you won't lose any heat due to conduction or convection, so the radiative heat loss is all you need to consider. Now, $T = 35 + 273 = 308$ K, so

$$\frac{dQ}{dt} = e\sigma A T^4 = 459 \text{ W}$$

Technically, you should subtract the absorption due to the cosmic microwave background, but that is negligible in comparison. What is not negligible is the 100 W generated by her metabolism, so

$$\frac{dQ_{\text{net}}}{dt} = 459 \text{ W} - 100 \text{ W} = 359 \text{ W}$$

This will turn out not to be too bad.

(c) Calculate the rate at which her body temperature will drop, which is $dT/dt \approx \Delta T/\Delta t$ for small time intervals.

Answer: Using $Q = mc\Delta T$ for temperature change,

$$359 \text{ W} = \frac{mc\Delta T}{\Delta t}$$

and so

$$\frac{dT}{dt} = \frac{359 \text{ W}}{(56.0 \text{ kg})(3400 \text{ J/kg·K})} = 1.89 \times 10^{-3} \text{ K/s}$$

This is slow.

(d) How long will it take for her to cool by 1 K?

Answer:

$$\frac{1\text{ K}}{1.89 \times 10^{-3}\text{ K/s}} = 530\text{ s} \approx 9\text{ minutes}$$

Freezing to death is not her immediate problem in this situation.