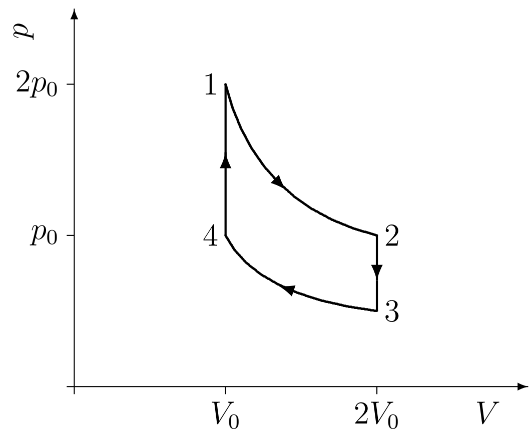
Solutions to Exam 3; Phys 185

1. (60 points) You have a monatomic ideal gas that goes through the cycle $1 \to 2 \to 3 \to 4 \to 1$ shown in the diagram. No gas molecules are added or removed during the cycle.

The $4 \to 1$ and $2 \to 3$ parts of the cycle at at constant volume. The $1 \to 2$ and $3 \to 4$ curves are such that the quantity $pV^{5/3}$ remains constant during these processes. You also know that $p_1 = 2p_0$, $V_1 = V_0$, $V_2 = 2V_0$ and $p_4 = p_0$, where p_0 and V_0 are constants.



(a) Find p_2 and p_3. Your answers should be numbers multiplied by p_0.

Answer: At point 1, $pV^{5/3} = 2p_0V_0^{5/3}$. Therefore, $p_2(2V_0)^{5/3} = 2p_0V_0^{5/3}$, leading to $p_2 = 2^{-2/3}p_0 = 0.63\,p_0$.

Similarly, $p_3(2V_0)^{5/3} = p_0V_0^{5/3}$ and $p_3 = 2^{-5/3}p_0 = 0.31\,p_0$.

(b) The area under the curve on the p-V diagram for curves where $pV^{5/3}$ is constant, going from from an initial p_i and V_i to a final p_f and V_f, is

$$-\tfrac{3}{2}p_iV_i^{5/3}\left(V_f^{-2/3} - V_i^{-2/3}\right)$$

Find the work done *on* the gas for each step of this cycle: $W_{1\to2}$, $W_{2\to3}$, $W_{3\to4}$, $W_{4\to1}$. Your answers should be numbers multiplied by p_0V_0.

Answer: The works are negatives of the areas under the curves.

$$W_{1\to2} = \tfrac{3}{2}2p_0V_0^{5/3}\left[(2V_0)^{-2/3} - V_0^{-2/3}\right] = 3(2^{-2/3} - 1)p_0V_0 = -1.11\,p_0V_0$$

$$W_{2\to3} = 0$$

$$W_{3\to4} = \tfrac{3}{2}p_0(2V_0)^{5/3}\left[V_0^{-2/3} - (2V_0)^{-2/3}\right] = \tfrac{3}{2}(1 - 2^{-2/3})p_0V_0 = 0.56\,p_0V_0$$

$$W_{4\to1} = 0$$

(c) Find the heats added to the gas: $Q_{1\to2}$, $Q_{2\to3}$, $Q_{3\to4}$, $Q_{4\to1}$. Your answers should be numbers multiplied by p_0V_0.

Answer: Use $Q = \Delta U - W$, so calculate the ΔU's first, with $U = \frac{3}{2}NkT = \frac{3}{2}pV$ for a monatomic ideal gas.

$$\Delta U_{1\to2} = U_2 - U_1 = \tfrac{3}{2}p_2(2V_0) - \tfrac{3}{2}(2p_0)V_0 = 3(2^{-2/3} - 1)p_0V_0 = -1.11\,p_0V_0$$
$$\Delta U_{2\to3} = U_3 - U_2 = \tfrac{3}{2}p_3(2V_0) - \tfrac{3}{2}p_2(2V_0) = 3(2^{-5/3} - 2^{-2/3})p_0V_0 = -0.94\,p_0V_0$$
$$\Delta U_{3\to4} = U_4 - U_3 = \tfrac{3}{2}p_0V_0 - \tfrac{3}{2}p_3(2V_0) = \tfrac{3}{2}(1 - 2^{-2/3})p_0V_0 = 0.56\,p_0V_0$$
$$\Delta U_{4\to1} = U_1 - U_4 = \tfrac{3}{2}(2p_0)V_0 - \tfrac{3}{2}p_0V_0 = \tfrac{3}{2}p_0V_0 = 1.5\,p_0V_0$$

Note that the total ΔU adds up to zero, as it should.

Now we subtract the W's from before:

$$Q_{1\to2} = \Delta U_{1\to2} - W_{1\to2} = 0$$
$$Q_{2\to3} = \Delta U_{2\to3} - W_{2\to3} = -0.94\,p_0V_0$$
$$Q_{3\to4} = \Delta U_{3\to4} - W_{3\to4} = 0$$
$$Q_{4\to1} = \Delta U_{4\to1} - W_{4\to1} = 1.5\,p_0V_0$$

(d) Do you notice anything special about the processes $1 \to 2$ and $3 \to 4$? Pick one of the following options for how to realize such a process in the lab, and explain your choice.

(i) Add (or remove) grains of sand to the top of the piston in exactly the right way to make sure that the pressure varies linearly with volume.

(ii) Heat (or cool) the gas in exactly the right way to make sure that the pressure varies exponentially with volume.

(iii) Bring the gas into contact with a heat reservoir, and expand or contract it very slowly, making sure its temperature stays constant.

(iv) Move the piston very quickly, giving the gas no time to have any thermal energy exchange with its environment. What is special about $1 \to 2$ and $3 \to 4$ is that $Q = 0$ for both. So you don't want any thermal energy transfer. If you do things very quickly, there won't be any time for any heat transfer.

(v) Keep the volume constant while varying the pressure exponentially.

(e) Find the total heat input to this gas in one cycle, Q_{in}. Also find the total heat removed from the gas, Q_{out}, and the total work done *by* the gas, W_{tot}.

Answer: Adding out the total positive heats, $Q_{\text{in}} = 1.5\,p_0V_0$. The negative heats are the exhaust: $Q_{\text{out}} = 0.94\,p_0V_0$.

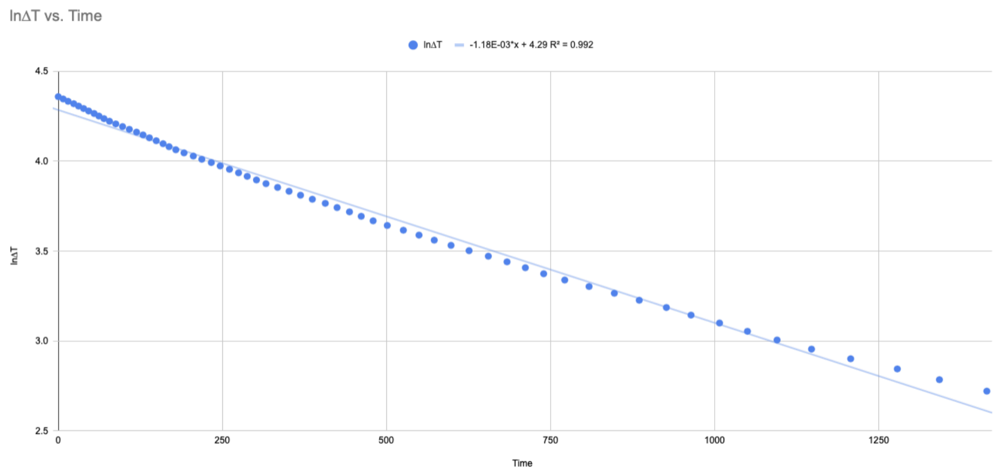
The total work done by the gas is the negative of the total work done on the gas: $W_{\text{tot}} = -(\frac{3}{2} - 3)(1 - 2^{-2/3})p_0V_0 = 0.56\,p_0V_0$.

(f) What is the efficiency of this heat engine? Your answer should be a number.

Answer:

$$e = \frac{W_{\text{tot}}}{Q_{\text{in}}} = 1 - 2^{-2/3} = 0.37$$

2. (40 points) Here is the graph of $\ln \Delta T$ vs t (with t in seconds) produced by one group with their data from Lab 8: Cooling.



Notice that while the straight line is a good fit, it isn't perfect. If only conduction and convection were operating, $\frac{dQ}{dt} = (\text{constants})\Delta T$. If you work out the math, you will get

$$\frac{d(\Delta T)}{dt} = -\frac{1}{\tau}\Delta T$$

Whenever you see an equation like this, the solution is an exponential:

$$\Delta T = \Delta T_0\, e^{-t/\tau} \quad \text{or} \quad \ln \Delta T = \ln \Delta T_0 - \frac{1}{\tau} t$$

which is a straight line graph above. So something else must also be going on: radiation, perhaps? After all, radiation depends on the difference of the *fourth powers* of the temperature of the aluminum cylinder and the temperature of the environment.

We can deal with radiation by using an approximation valid when ΔT is small:

$$\Delta(T^4) \approx 4T^3 \Delta T$$

(a) To use such an approximation, what must ΔT be small compared to? Explain your choice.

(i) T**.** The only one that has the correct units (you can only compare a temperature to a temperature), and the only one that makes any sense.

(ii) $t/\Delta T_0$

(iii) τ

(iv) $\ln T$

(v) $\ln(t/\Delta T_0)$

(vi) $\ln \tau$

(b) Call the temperature of the aluminum cylinder when it first starts cooling down T_{begin}, and the temperature when you're just about to stop collecting data, T_{end}. Follow the procedure to find the equation for $\ln \Delta T$ above, but do it for radiation: find τ_{begin} and τ_{end}. Determine whether $\tau_{\mathsf{begin}} > \tau_{\mathsf{end}}$, $\tau_{\mathsf{begin}} = \tau_{\mathsf{end}}$, or $\tau_{\mathsf{begin}} < \tau_{\mathsf{end}}$.

Hint 1: You'll need $Q = mc\Delta T$. But the ΔT in this equation is ΔT_{cyl}, the change in temperature of the cylinder, not ΔT_{env}, the difference between the cylinder and environment temperatures. Fortunately, as long as the environment doesn't change, their rates of change are the same: $d(\Delta T_{\mathsf{cyl}})/dt = d(\Delta T_{\mathsf{env}})/dt$.

Hint 2: Say A is a constant, while B is a quantity that changes in time. Then, the rate of change $d(AB)/dt = A\, dB/dt$.

Answer: There is radiation absorbed from the environment (gain), and radiation emitted away from the aluminum cylinder (loss). The total heat transfer rate to the cylinder is

$$\frac{dQ}{dt} = e\sigma A(T^4_{\mathsf{environment}} - T^4_{\mathsf{cylinder}}) = -e\sigma A\Delta(T^4) \approx -4T^3 e\sigma A\Delta T$$

Notice that the only thing here that is different is the T^3 factor. We then use $Q = mc\Delta T_{\mathsf{cyl}}$:

$$\frac{dQ}{dt} = mc\frac{d(\Delta T_{\mathsf{cyl}})}{dt} = mc\frac{d(\Delta T)}{dt}$$

Putting them all together, we have

$$\frac{d(\Delta T)}{dt} = -\left(\frac{4T^3 e\sigma A}{mc}\right)\Delta T$$

The quantity in brackets is $1/\tau$:

$$\tau = \frac{mc}{4T^3 e\sigma A} \quad \Rightarrow \quad \tau_{\mathsf{begin}} = \frac{mc}{4T^3_{\mathsf{begin}} e\sigma A} \quad \text{and} \quad \tau_{\mathsf{end}} = \frac{mc}{4T^3_{\mathsf{end}} e\sigma A}$$

Since $T_{\mathsf{begin}} > T_{\mathsf{end}}$, and everything else is the same, $\tau_{\mathsf{begin}} < \tau_{\mathsf{end}}$.

(c) Given your answer to (b), give an argument for whether the deviation from linearity seen in the graph of the experimental data is consistent or inconsistent with radiation being a noticeable component of the heat transfer from the aluminum cylinder.

Answer: A small τ corresponds to a steep negative slope on the graph of the Lab 8 data: the cooling is rapid. Since $\tau_{\mathsf{begin}} < \tau_{\mathsf{end}}$, that means the contribution of radiation to the initial cooling is strong, and this contribution diminishes as the cylinder becomes colder. So due to radiation, we should see more rapid cooling at the beginning, and a more steeply negative slope on the $\ln \Delta T$ vs t graph, and as time passes, the slope should become less steeply negative. That is exactly what the graph shows.