Solutions to Assignment 3; Phys 186

1. (20 points) A large refractive index lens on eyeglasses has $n \approx 1.7$, compared to about 1.5 for ordinary glass. Imagine that you read a website that says:

> You will never find glasses with lenses using a material with $n \geq 2$ due to total internal reflection. For such a large index of refraction, the critical angle beyond which light rays from air cannot enter the glass and are totally reflected back is relatively small. This severely restricts peripheral vision, and thus such materials are not suitable to make lenses for eyeglasses.

Does this seem correct, or is it yet another example of the sort of nonsense you can find on the web? Whatever your answer is, support it with a *quantitative* argument.

Answer: This is nonsense. You cannot have total internal reflection for a light ray going from a small index of refraction medium (air) into a larger index of refraction medium (eyeglasses). Calculate the critical angle, for $\theta_2 = 90^\circ$ in $n_1 \sin\theta_1 = n_2 \sin\theta_2$. This gives a critical angle

$$\theta_1 = \sin^{-1}\frac{n_2}{n_1} = \sin^{-1} 2$$

There is no angle with a sine of 2 or larger. Every light ray from the air will be able to enter the eyeglass lens.

2. (30 points) You pass white light through a prism, and see the ray of light split into a rainbow pattern. Choose between four possible explanations:

(a) The prism is a diffraction grating, with the spaces between atoms acting as slits.

(b) The prism is a double slit, with the spaces between atoms acting as slits.

(c) The prism has an index of refraction n that depends on the wavelength λ, with n increasing as λ increases.

(d) The prism has an index of refraction n that depends on the wavelength λ, with n decreasing as λ increases.

Explain how each of the following is relevant to your decision:

- Atoms in a crystal such as the prism are separated by distances around 1 nm.

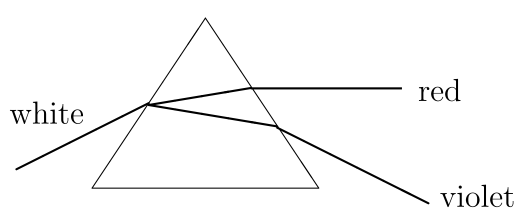
  Answer: Since visible light has a wavelength about 500 nm, a diffraction grating or double slit with a slit separation of 1 nm won't work very well to give a rainbow pattern. After all, $\sin\theta = m\,\lambda/d$, and $\lambda/d \approx 500$—the $m = \pm 1$ peak does not exist. Another way of putting it is that $d \ll \lambda$, so you won't see much of wave-like effects such as interference to produce the rainbow.

- No white light gets through; only the rainbow pattern is seen.

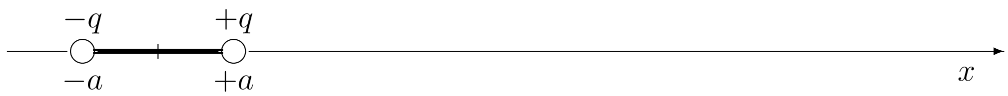
  Answer: If the prism was acting as a double slit or a diffraction grating, you'd also see a strong central maximum of white light. You don't.

- Red light appears at the top, violet at the bottom:

  Answer: You can see that red light ray (long wavelength λ) is bent less than the violet (short wavelength λ). The larger the index of refraction n, the stronger the refraction you will observe. Therefore n must depend on λ and n must *decrease* with increasing λ.



3. (50 points) You have an electric dipole arranged on the x-axis: a $+q$ charge at $x = +a$ and a $-q$ charge at $x = -a$. The charges are connected by a rigid rod, so the distance between them never changes.



(a) Calculate the electric field created by this dipole on a point on the x-axis, for $x > a$. Get both magnitude and direction.

Answer: At $x > a$, the distance to $+q$ is $x - a$ and the distance to $-q$ is $x + a$. The magnitudes of the electric field due to the two charges are

$$E_+ = \frac{kq}{(x-a)^2} \qquad E_- = \frac{kq}{(x+a)^2}$$

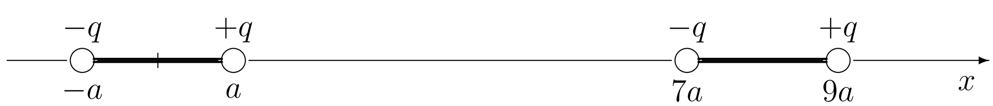
$E_+ > E_-$, because $+q$ is closer to points where $x > a$.

Note that $\vec{E}_+$ points in the $+x$ direction, and $\vec{E}_-$ points in the $-x$ direction. Both electric fields are along the x-axis, with no y component. So we can add them together without worrying about trigonometry. The total electric field is

$$E = E_+ - E_- = kq\left[\frac{1}{(x-a)^2} - \frac{1}{(x+a)^2}\right]$$

The direction of $\vec{E}$ for $x > a$ is in the $+x$ direction.

(b) Calculate the force the first dipole exerts on another dipole further down the x-axis. Get both magnitude and direction.



Answer: Use the electric field found for $x = 7a$ and $x = 9a$, and multiply by the charges at those locations:

$$F_{7a} = kq(-q)\left[\frac{1}{(6a)^2} - \frac{1}{(8a)^2}\right] = -0.012\,\frac{kq^2}{a^2}$$

$$F_{9a} = kq(+q)\left[\frac{1}{(8a)^2} - \frac{1}{(10a)^2}\right] = 0.0056\,\frac{kq^2}{a^2}$$

The total force is

$$F = -0.0065\,\frac{kq^2}{a^2}$$

The $-$ sign indicates a force in the $-x$ direction: the second dipole is attracted by the first.

(c) You now have the second dipole oriented perpendicular to the first:

Qualitatively sketch the forces on this second dipole due to the first dipole. Also indicate in what direction (clockwise or counterclockwise) it rotates. Very briefly explain why.

Answer: The $-q$ on the second dipole will be attracted toward the first dipole, because the $+q$ on the first dipole is closer. The $+q$ on the second dipole will be repelled away from the first dipole, again because the $+q$ on the first dipole is closer. This pair of forces will produce a counterclockwise rotating torque. So the second dipole will tend to align with the first one.

(d) What can you conclude about dipole-dipole interactions from this problem? Do dipoles attract or repel one another? How do they orient themselves relative to each other?

Answer: Dipoles attract, and they orient themselves such that arrows you draw from the $-$ to the $+$ charges in each dipole are parallel to one another.

Extra Problems (not graded)

4. (0 points) If you dive down under water on a bright sunny day and you look up, you will see the whole sky within a circle on top of you, with darkness on the sides.

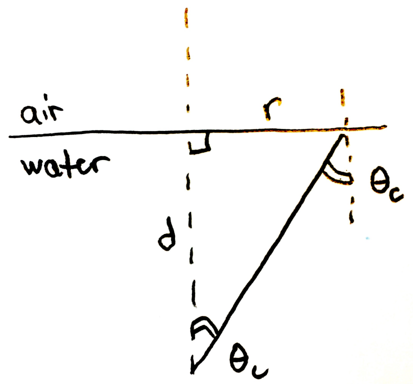
(a) Explain what is happening. (Looking up an online explanation is fine, but make sure what you write here is *only* in terms of what we discussed in class.)

Answer: Since the index of refraction of water is larger than that of air, the direction of propagation of light waves will bend closer to the vertical once they enter the water. All of the sky means an entrance angle θ_1 of 0 to 90° with the vertical. In the water, the direction of propagation will go as $\theta_2 = \sin^{-1}(\sin\theta_1/n_2)$, from 0 to a critical angle $\theta_c < 90°$. So all the light from the sky will come down in a cone defined by θ_c, and you will see a circle of light above.

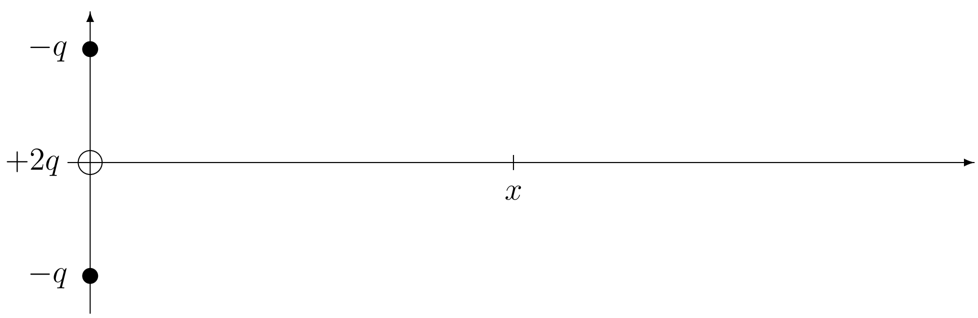
(b) Let's say you're at a depth d below the surface. Express the radius r of the circle of light in an equation that might depend on d and anything else you think is necessary.

Answer: If the light is coming down in a cone with angle $\theta_c < 90°$, the relationship of r to d will be that $r/d = \tan\theta_c$. Looking up the index of refraction of water, $n_2 = 1.333$, we get

$$r = \left(\tan\sin^{-1}\frac{1}{1.333}\right) d = 1.13\, d$$



5. (0 points) You have charges $-q$ at $x = 0, y = a$ and $x = 0, y = -a$, and a charge $+2q$ at $x = 0, y = 0$. The positions of the charges are fixed.



(a) Find the total electric field at a point a distance x from the origin on the x axis. In other words, find the total E_x and E_y as functions of k, q, a, and x.

Answer: Label the charges 1, 2, 3, from the top. The magnitudes of the electric fields are

$$E_1 = E_3 = \frac{kq}{a^2 + x^2} \qquad E_2 = \frac{2kq}{x^2}$$

From the symmetry of the set-up, you can see that the y-components of $\vec{E}_1$ and $\vec{E}_3$ will cancel each other out, leaving a total $E_y = 0$. The x-components are, when we include the factors of $\cos\theta = x/\sqrt{a^2 + x^2}$,

$$E_{1x} = E_{3x} = -\frac{kqx}{(a^2 + x^2)^{3/2}} \qquad E_{2x} = \frac{2kq}{x^2}$$

Adding all these up,

$$E_x = 2kq\left(\frac{1}{x^2} - \frac{x}{(a^2+x^2)^{3/2}}\right)$$

(b) If you can cast your previous answer in a form that puts all the a-dependence in a term that goes like $(1 + a^2/x^2)^{-3/2}$, you can use the approximation

$$\left(1+\frac{a^2}{x^2}\right)^{-3/2} \approx 1 - \frac{3a^2}{2x^2}$$

valid for $a \ll x$. Use this to show that for large x values, the electric field goes like $E \propto x^{-4}$, an inverse fourth power law. (Ask me for math help if you need it.)

Answer: Rewrite the previous answer as

$$E_x = \frac{2kq}{x^2}\left[1 - \left(1+\frac{a^2}{x^2}\right)^{-3/2}\right]$$

Use the approximation:

$$E_x \approx \frac{2kq}{x^2}\left[1-\left(1-\frac{3a^2}{2x^2}\right)\right] = \frac{3kqa^2}{x^4} \propto x^{-4}$$

By the way, this is a variety of *quadrupole* charge arrangement. Notice that its electric field drops off with distance even faster than with a dipole.