Homework Solutions 1 (Griffiths Chapter 1)

In the following, I'll have the solid angle element $d\Omega = sin\theta \, d\theta \, d\phi$. And therefore, standing alone, $\oint d\Omega = 4\pi$, integrated over the full solid angle.

39 We want to check $\int_V dv \, \nabla \cdot \mathbf{v} = \oint_{\partial V} d\mathbf{a} \cdot \mathbf{v}$. For our spherical volume, $d\mathbf{a} = d\Omega R^2 \hat{\mathbf{r}}$.

$$\int_V dv \, \nabla \cdot \mathbf{v_1} = \int_V dv \, \frac{1}{r^2} \frac{\partial}{\partial r} r^4 = \oint d\Omega \int_0^R dr \, r^2 \, (4r) = 4\pi R^4$$

$$\oint_{\partial V} d\mathbf{a} \cdot \mathbf{v_1} = \oint d\Omega \, R^4 = 4\pi R^4$$

These are obviously equal.

Now, it may seem that

$$\int_V dv \, \nabla \cdot \mathbf{v_2} = \int_V dv \, \frac{1}{r^2} \frac{\partial}{\partial r} 1 = 0$$

but this is misleading, since there are infinities involved when $r \to 0$. Looking at the surface integral

$$\oint_{\partial V} d\mathbf{a} \cdot \mathbf{v_2} = \oint d\Omega \, 1 = 4\pi$$

Since this integral gives 4π for all $R > 0$, this must mean that $\nabla \cdot \mathbf{v_2} = 0$ for all $r > 0$, but when we include $r = 0$, the integral is 4π. Those properties define the three-dimesional delta function, so

$$\nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2} \right) = 4\pi \, \delta^3(\mathbf{r})$$

45 Change variables in each case:

(a) With $u = 3x$ and $du = 3dx$, the integral becomes

$$\int_{-6}^{6} \frac{du}{3} \, \delta(u) \left(\frac{2}{3} u + 3 \right) = 1$$

(b) $u = 1 - x$, $du = -dx$

$$\int_{-1}^{1} du\,\delta(u)\left[(1-u)^3 + 3(1-u) + 2\right] = 6$$

(c) $u = 3x + 1$, $du = 3dx$

$$\int_{-1}^{4} \frac{du}{3}\,\delta(u)\,9\left(\frac{u-1}{3}\right)^2 = \frac{1}{3}$$

(d) $\int_{-\infty}^{a} dx\,\delta(x-b) = 0$ if $b > a$ and $= 1$ if $b < a$. The result is ambiguous if $b = a$.

49 First, use the delta function result from problem (39):

$$\int_V dv\,e^{-r}\left[\boldsymbol{\nabla}\cdot\left(\frac{\hat{\mathbf{r}}}{r^2}\right)\right] = \int_V dv\,e^{-r}\delta^3(\mathbf{r}) = 4\pi$$

Then, integrating by parts,

$$\int_V dv\,e^{-r}\left[\boldsymbol{\nabla}\cdot\left(\frac{\hat{\mathbf{r}}}{r^2}\right)\right] = \oint_{\partial V} d\mathbf{a}\cdot\left(\frac{\hat{\mathbf{r}}}{r^2}\right)e^{-r} - \int_V dv\left(\frac{\hat{\mathbf{r}}}{r^2}\right)\cdot\boldsymbol{\nabla}(e^{-r}) =$$

$$\oint d\Omega e^{-R} + \oint d\Omega\int_0^R r^2 dr\,\frac{1}{r^2}e^{-r} = 4\pi\,e^{-R} - 4\pi\,e^{-R} + 4\pi = 4\pi$$

50 Divergences and curls:

$$\boldsymbol{\nabla}\cdot\mathbf{F}_1 = 0 \qquad \boldsymbol{\nabla}\times\mathbf{F}_1 = -\frac{\partial}{\partial x}F_{1z}\hat{\mathbf{y}} = -2x\hat{\mathbf{y}}$$

$$\boldsymbol{\nabla}\cdot\mathbf{F}_2 = 3 \qquad \boldsymbol{\nabla}\times\mathbf{F}_2 = 0$$

Since $\boldsymbol{\nabla}\times\boldsymbol{\nabla}\phi = 0$, we can have $\mathbf{F}_2 = \boldsymbol{\nabla}\phi$, where $\phi = \frac{1}{2}(x^2+y^2+z^2)$ is one possibility for ϕ.

Since $\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\times\mathbf{A} = 0$, we can have $\mathbf{F}_1 = \boldsymbol{\nabla}\times\mathbf{A}$, where $\mathbf{A} = \frac{1}{3}x^3\hat{\mathbf{y}}$ is one possibility for $\mathbf{A}$.

Then, note that $\boldsymbol{\nabla}\cdot\mathbf{F}_3 = \boldsymbol{\nabla}\times\mathbf{F}_3 = 0$. Therefore we can write

$$\mathbf{F}_3 = \boldsymbol{\nabla}\phi, \text{ with } \phi = xyz$$

$$\mathbf{F}_3 = \boldsymbol{\nabla} \times \mathbf{A}, \text{ with } \mathbf{A} = \frac{1}{4}\left[x(y^2 - z^2)\hat{\mathbf{x}} + y(z^2 - x^2)\hat{\mathbf{y}} + z(x^2 - y^2)\hat{\mathbf{z}}\right]$$

63 First, the general case:

$$\boldsymbol{\nabla} \cdot (r^n \hat{\mathbf{r}}) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^{n+2}\right) = (n+2)r^{n-1} \text{ for } n > -2$$

For $n = -1$, this gives $\boldsymbol{\nabla} \cdot \mathbf{v} = 1/r^2$. Integrating this over a sphere centered on the origin,

$$\int_V dv\, \boldsymbol{\nabla} \cdot \mathbf{v} = \oint d\Omega \int_0^R dr = 4\pi R$$

Using the divergence theorem,

$$\oint_{\partial V} d\mathbf{a} \cdot \frac{\hat{\mathbf{r}}}{r} = \oint d\Omega R = 4\pi R$$

These are the same, so no δ-functions are involved.

Using the spherical form for the curl,

$$\boldsymbol{\nabla} \times (r^n \hat{\mathbf{r}}) = 0$$

since all the θ and ϕ components of the vector field are zero, and the r-component does not depend on θ or ϕ. Using the result of problem (61b),

$$\oint_{\partial V} d\mathbf{a} \times (r^n \hat{\mathbf{r}}) = \oint_{\partial V} da\, r^n (\hat{\mathbf{r}} \times \hat{\mathbf{r}}) = 0$$