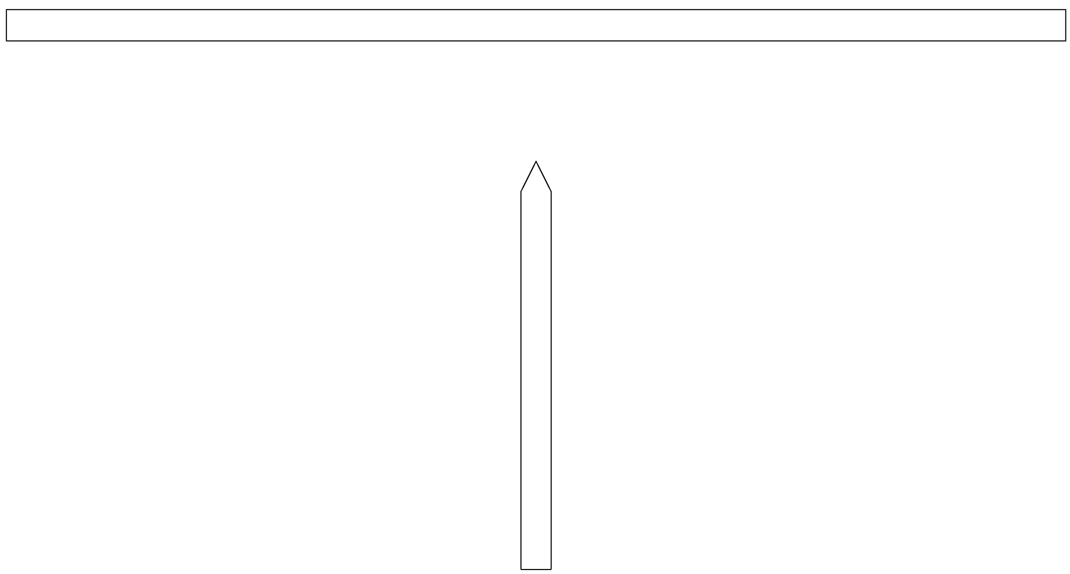
Solutions to Exam 2; Phys 186

1. (20 points) You have an arrangement with a flat metal plate at a 5.0 V voltage, and a sharp metal pin at 0.0 V, oriented perpendicular to the plate.



(a) Draw equipotential lines in 1 V intervals. And then draw in electric field lines.

Answer: The metal surfaces are themselves equipotential lines at 5.0 V and 0.0 V. The lines in between gradually change from one surface to another, which means that the equipotential lines get bunched up where the sharp point gets close to the plate, but get more widely separated off to the sides. The electric field lines will be perpendicular to the equipotential lines, directed from the plate (higher voltage) toward the pin (lower voltage). To be consistent, the separation between the electric field lines will also be smaller closer to the middle where the pin comes close to the plate.

(b) Draw + and − charges inside the metals to indicate the charge distribution.

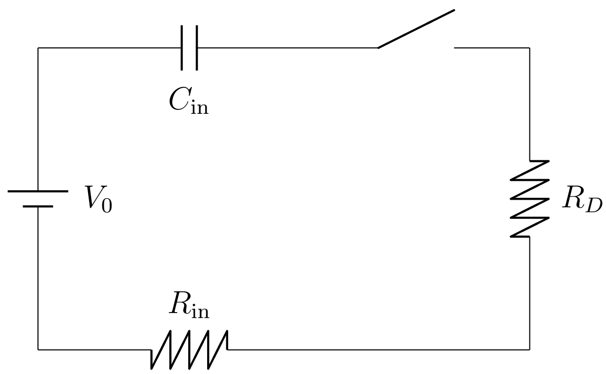
Answer: The + charges will go on the plate, and they will closer together around where the pin comes close. The − charges will go on the pin, and will cluster closer together toward the tip.

(c) Where is the electric field the strongest? Lightning strikes are electrical discharges located where the electric field is strongest. Can you now see why lightning rods are long and pointed?

Answer: The electric field is strongest where both the equipotential lines and electric filed lines are denser. In this case, that is the area where the pin is closest to the plate. Lightning rods, by being sharp rods, also will have electric fields strongest closest to their tips, and therefore will provide a more likely path for lightning discharges.

2. (40 points) We discussed how real batteries had some internal resistance R_{in}, which limited the current you can draw from them. Let's now explore what would happen if batteries also had an internal capacitance C_{in}, and you connected your more realistic model of a battery that supplies V_0 to a device with resistance R_D.

(a) Let's say C_{in} was in series with R_{in} and the ideal battery. The capacitor has no charge before the switch is closed. Figure out I_D, the current through the device, immediately after the switch is closed and a long time after the switch is closed. Then sketch a graph of I_D vs. t. What does the presence of a capacitor do in this model of a battery: introduce a time delay, waste extra energy, make the battery useless, or increase the current delivered to the device?



Answer: The loop equation gives

$$V_0 = V_C + V_D + V_{\text{in}} = \frac{Q}{C_{\text{in}}} + R_D I + R_{\text{in}} I$$

Immediately after the switch is closed, $Q = 0$, therefore

$$I_D(0) = \frac{V_0}{R_D + R_{\text{in}}}$$

After a long time, the current though the capacitor, and therefore the whole circuit, becomes zero:

$$I_D(\infty) = 0$$

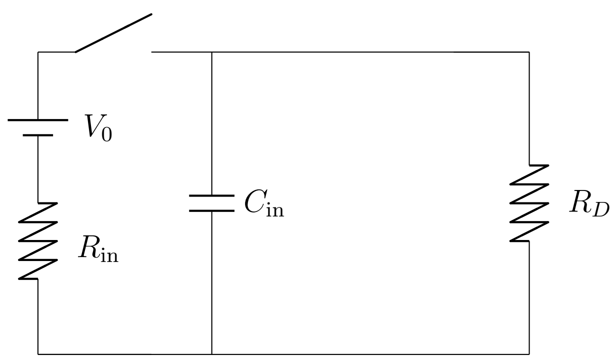
This is a circuit for charging up the capacitor, so actually

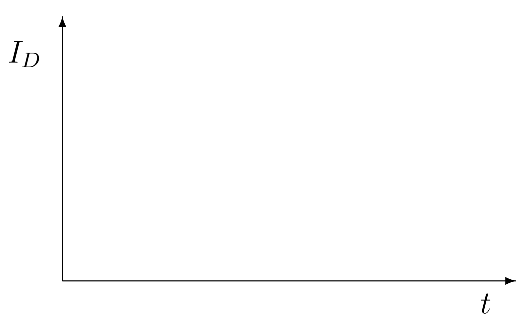
$$I_D(t) = \frac{V_0}{R_D + R_{\text{in}}} e^{t/(R_D + R_{\text{in}})C}$$

The graph will be a typical current graph for a capacitor charging up: an exponential decay.

Note that since the current will go down to zero, this renders the battery useless for long times. In other words, this is not a good model for including the internal capacitance of a battery.

(b) Let's say C_{in} was in parallel with R_{in} and the ideal battery. The capacitor has no charge before the switch is closed. Figure out I_D, the current through the device, immediately after the switch is closed and a long time after the switch is closed. Then sketch a graph of I_D vs. t. What does the presence of a capacitor do in this model of a battery: introduce a time delay, waste extra energy, make the battery useless, or increase the current delivered to the device?





Answer: The two loop equations are

$$V_0 = \frac{Q}{C_{\text{in}}} + R_{\text{in}} I_0 \qquad \text{and} \qquad \frac{Q}{C_{\text{in}}} = R_D I_D$$

The junction equation is

$$I_0 = I_C + I_D$$

Immediately after the switch is closed, $Q = 0$ and therefore there will be no current through the device,

$$I_D(0) = 0$$

After a long time, the current though the capacitor becomes zero, so $I_0 = I_D$ and

$$I_D(\infty) = \frac{V_0}{R_D + R_{\text{in}}}$$

This is a circuit identical to Assignment 5, question (2a). The current will be

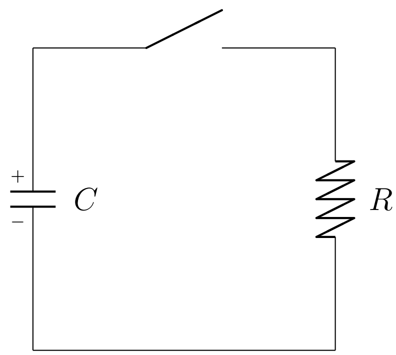
$$I_D(t) = \frac{V_0}{R_D + R_{\text{in}}}\left(1 - e^{t/R_{\text{in}}C}\right)$$

The current will rise toward and approach a maximum value. The presence of a capacitor introduces a short time delay, which is realistic enough.

3. (40 points) You have a capacitor discharge circuit: The capacitor C starts *fully charged* before the switch is closed, with a voltage V_0 across it, with the $+Q$ and $-Q$ plates as indicated. Once the switch is closed, the capacitor starts discharging, and there will be a current I through the circuit. That current produces a magnetic field through the circuit itself; call the average of the perpendicular component of the magnetic field within the circuit $\bar{B}_\perp = kI$, where k is a constant. The circuit is a square, with sides of length a each.

(a) Find the equation for the induced voltage $V_{\text{ind}}(t)$. Then sketch a rough graph of how $V_{\text{ind}}(t)$ depends on t.

Hint: The rate of change of an exponential: $\frac{d}{dt}e^{-t/\tau} = -\frac{1}{\tau}e^{-t/\tau}$.



Answer: The magnetic flux though the circuit is

$$\Phi_B = \bar{B}_\perp A = kIa^2$$

The current through a discharging capacitor in this circuit is

$$I = \frac{V_0}{R}\, e^{-t/RC}$$

The induced voltage will be

$$\begin{aligned} V_{\text{ind}} &= -\frac{d}{dt}\Phi_B = -\frac{d}{dt}\left(ka^2\frac{V_0}{R}\, e^{-t/RC}\right) = -\left(ka^2\frac{V_0}{R}\right)\frac{d}{dt}\, e^{-t/RC} \\ &= \left(ka^2\frac{V_0}{R}\right)\frac{1}{RC}\, e^{-t/RC} = \frac{ka^2V_0}{R^2C}\, e^{-t/RC} \end{aligned}$$

The graph will be an exponential decay.

(b) Imagine you now talk to a senior physics major, and she points out to you that the extra voltage induced in the circuit will mean an extra current on top of what you accounted for, and therefore an extra magnetic flux. Since you ignored that extra magnetic flux, your calculation in part (a) is, strictly speaking, incorrect!

You still can use your result as a very good approximation, however, if the induced voltage is small: if $V_{\text{ind}}/V_C \ll 1$, where V_C is the voltage across the capacitor, then you have nothing to worry about. Find an equation for this ratio V_{ind}/V_C.

$$\frac{V_{\text{ind}}}{V_C} = \frac{\frac{ka^2 V_0}{R^2 C}\, e^{-t/RC}}{V_0\, e^{-t/RC}} = \frac{ka^2}{R^2 C}$$

Then, to see if the ratio is small enough, choose from among the following options and circle your choices:

- Which is a reasonable value for the size of the circuit in the lab, a?

  10.0 m   1.00 m $\underline{0.10\,\text{m}}$

- Which is a reasonable value for a resistance in the lab, R?

  1.0 MΩ $\underline{1.0\,\Omega}$ 1.0 μΩ

- What is a typical value for capacitances you have worked with in the lab, C?

  0.01 MF   0.01 F $\underline{0.01\,\mu\text{F}}$

- The only information you can find about the magnetic field through a current loop is $B_\perp = \mu_0 I/2r$, which refers to the magnetic field at the center of a circular current loop with radius r. That will have to do; all you need is a rough estimate. So, what is your estimate for k in this situation?

$$k = \frac{\mu_0}{2r} \text{ with } r \approx 0.1/2\,\text{m} = 1.26 \times 10^{-5}\,\text{kg/C}^2$$

Now put all the numbers together and find a numerical value for V_{ind}/V_C. Will your result from (a) be useful in normal lab conditions, or will it not?

Answer: We get

$$\frac{ka^2}{R^2C} = 12.6 > 1$$

This means that the approximation we made was *not* good! The time scale RC is too small, so the changes happen too quickly: the rate of change of the magnetic flux is too large, larger than the capacitor voltage. So more advanced physics is required to calculate what exactly is going on.