1. (100 points) You have a sphere with radius R and a surface charge distribution $\sigma = k \sin\theta \sin\phi$.

(a) Write down the 3D charge distribution $\rho(r, \theta, \phi)$. Later, if you need to do any integrals involving ρ, use your answer here.

Answer: This is trivial: $\rho = k \sin\theta \sin\phi\, \delta(r - R)$.

(b) Find $\mathbf{E}(\mathbf{r})$, the electric field everywhere in space. *Hint:* one way to do this involves the spherical harmonics $Y_l^m(\theta, \phi)$. If you do so, don't reinvent the wheel: feel free to look up whatever you need to know about spherical harmonics.

Answer: Doing a full-blown separation of variables for $\nabla^2 V = 0$, we found that

$$V = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left(A_{lm} r^l + B_{lm} r^{-(l+1)} \right) Y_l^m(\theta, \phi)$$

(If your notes are incomplete, you can find this by looking it up online or in many textbooks.)

Outside of the sphere, to avoid V blowing up as $r \to \infty$, we must have $A_{lm} = 0$ for all l, m. Similarly, inside the sphere, to avoid blow-ups at $r = 0$, we must have $B_{lm} = 0$ for all l, m.

V is continuous at $r = R$, so $A_{lm} R^l = B_{lm} R^{-(l+1)}$, or $B_{lm} = A_{lm} R^{2l+1}$.

The normal derivative of the voltage is discontinuous:

$$\left[\frac{\partial V_{\text{in}}}{\partial r} - \frac{\partial V_{\text{out}}}{\partial r} \right]_{r=R} = \frac{\sigma}{\epsilon_0}$$

$$\sum_{lm} (2l + 1) A_{lm} R^{l-1}\, Y_l^m(\theta, \phi) = \frac{k}{\epsilon_0} \sin\theta \sin\phi$$

Now, looking at a table of spherical harmonics, you might notice that you can write

$$\sin\theta \sin\phi = \frac{i}{2} \sqrt{\frac{8\pi}{3}} \left(Y_1^1 + Y_1^{-1} \right)$$

The orthogonality of the spherical harmonics, $\int d\Omega\, Y_l^{m*} Y_{l'}^{m'} = \delta_{ll'}\delta_{mm'}$, means that the only non-zero coefficients are A_{11} and A_{1-1}:

$$A_{11} = A_{1-1} = \frac{i}{6}\sqrt{\frac{8\pi}{3}}\frac{k}{\epsilon_0}$$

Therefore

$$V_{\rm in}(\mathbf{r}) = A_{11} r \left(Y_1^1 + Y_1^{-1}\right) = \frac{kr}{6i\epsilon_0}\sin\theta\left(e^{i\phi} - e^{-i\phi}\right) = \frac{kr}{3\epsilon_0}\sin\theta\sin\phi$$

$$V_{\rm out}(\mathbf{r}) = A_{11}\frac{R^3}{r^2}\left(Y_1^1 + Y_1^{-1}\right) = \frac{kR^3}{3\epsilon_0 r^2}\sin\theta\sin\phi$$

Notice that the outside voltage is a pure dipole, $V \propto 1/r^2$.

Finally, the electric field is

$$\begin{aligned}\mathbf{E}_{\rm in}(\mathbf{r}) &= -\nabla V_{\rm in} = -\frac{k}{3\epsilon_0}\left(\sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}}\right) \\ &= -\frac{k}{3\epsilon_0}\,\hat{\mathbf{y}}\end{aligned}$$

$$\begin{aligned}\mathbf{E}_{\rm out}(\mathbf{r}) &= -\frac{kR^3}{3\epsilon_0 r^3}\left(-2\sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}}\right) \\ &= -\frac{kR^3}{3\epsilon_0 r^3}\left(\hat{\mathbf{y}} - 3\frac{y}{r}\hat{\mathbf{r}}\right)\end{aligned}$$

This is the electric field of a dipole along the y-axis.

(c) Find the monopole, dipole and quadrupole moments of this charge distribution. *Hint:* By comparing the complete answer you found in (b) with what is produced by your monopole and dipole moments, you may be able to avoid calculating the quadrupole integrals.

Answer: The monopole moment:

$$Q = \int dv'\, k\sin\theta'\sin\phi'\,\delta(r' - R) = 0$$

The dipole moment is a vector.

$$\mathbf{p} = \int dv'\, \mathbf{r}'\, k \sin\theta' \sin\phi'\, \delta(r' - R)$$

Cartesian coordinates ends up best when integrating. Use $\mathbf{r}' = r'\hat{\mathbf{r}}'$ and $\hat{\mathbf{r}}' = \sin\theta' \cos\phi'\, \hat{\mathbf{x}} + \sin\theta' \sin\phi'\, \hat{\mathbf{y}} + \cos\theta'\, \hat{\mathbf{z}}$. The result is

$$\mathbf{p} = \frac{4\pi}{3} k R^3\, \hat{\mathbf{y}}$$

Again, a dipole along the y-axis. We already found, in part (b), that the outside voltage and field was that of a pure dipole. Therefore, there cannot be any non-zero higher-order multipoles.