Solutions to Assignment 7; Phys 186

1. (30 points) An important discovery in the 1980s was the W^+ and W^-, which are among the particles responsible for the weak nuclear force. W^+ and W^- are antiparticles of each other, and they each have a rest mass of 80.4 GeV/c^2. Say you want to create a W^+ and W^- pair by a head-on collision of an electron and a positron (e^- and e^+) into each other at speeds close to the speed of light. The rest mass of an electron (and a positron, its antiparticle) is 0.511 MeV/c^2. (Remember: 1 GeV = 1000 MeV.)

(a) As observed from the lab frame of reference, the e^- and e^+ head toward each other with equal and opposite velocities in the collision. What is the minimum time dilation factor γ that the e^- and e^+ must have in order to produce enough energy to create a W^+ and W^- pair at rest?

Answer: Energy conservation means that

$$2m_W c^2 = 2\gamma m_e c^2 \qquad \Rightarrow \qquad \gamma = \frac{m_W}{m_e} = 1.57 \times 10^5$$

This is an enormous γ; the speed of the electron must be almost c.

(b) Say that in the lab frame of reference, the electron traveled 30.0 km at a constant speed corresponding to the γ you calculated in (a). How far did it travel in its own frame of reference?

Answer: In the lab frame, the 30.0 km distance is at rest; therefore, the proper length $\Delta x_0 = 30.0$ km. This distance in the electron's frame is contracted, as it is moving: the endpoint of the electron's flight is rushing at very high speed toward the stationary electron. Length contraction: $\Delta x = \Delta x_0/\gamma$, therefore

$$\Delta x = \frac{3 \times 10^4 \text{ m}}{1.57 \times 10^5} = 0.191 \text{ m}$$

(c) Calculate how long it took for the electron to travel 30.0 km in the lab frame of reference. Then calculate how long this time interval was in the electrons own frame of reference.

Answer: The speed is almost c. Therefore, to travel 30 km,

$$\Delta t = \frac{\Delta x_0}{c} = \frac{3 \times 10^4 \text{ m}}{3 \times 10^8 \text{ m/s}} = 10^{-4} \text{ s}$$

Note that this is *not* the proper time: in the lab frame, the beginning and end of the electron's flight do not take place at the same location. In the electron's frame, however, since the endpoint is rushing toward a stationary electron, the beginning and ending events happen at the same location. So the electron will have the proper time,

$$\Delta t_0 = \frac{\Delta t}{\gamma} = \frac{\Delta x}{c} = 6.37 \times 10^{-10} \text{ s}$$

2. (40 points) Say you're doing the lab where you accelerated and shot a beam of electrons onto a screen. The mass of an electron is $m_e = 511\,\text{keV}/c^2$.

(a) You accelerated the electrons through a voltage difference of up to 5.00 kV on your dial. At $V_a = 5.00\,\text{kV}$, then, what is the kinetic energy of the electrons in the beam, in units of keV? *Hint:* 1 eV is literally the electron charge magnitude e multiplied by 1 V. Therefore, you shouldn't need any real calculation to get this answer.

Answer: $e(5\,\text{kV}) = 5\,\text{keV}$.

(b) What fraction of the speed of light are these electrons traveling? Use $\frac{1}{2}m_e v^2$ for your kinetic energy, as in your homework and the lab.

Answer: The usual energy conservation gives

$$5\,\text{keV} = \frac{1}{2}(511\,\text{keV})\frac{v^2}{c^2} \quad \Rightarrow \quad \frac{v}{c} = \sqrt{\frac{2 \cdot 5}{511}} = 0.140$$

14% of the speed of light. I mentioned that our electrons in the lab were fast.

(c) Recalculate the fraction of the speed of light the electron has, using a more appropriate expression for kinetic energy.

Answer: A more accurate calculation of the speed would go like this. The total energy of the electron is $\gamma m_e c^2$, while the energy of an electron at rest, where $\gamma = 1$, is $m_e c^2$. Therefore, the kinetic energy, which is the additional energy an object has due its motion, must be the difference: $K = (\gamma - 1)m_e c^2$. We can use this relativistic form of kinetic energy to recalculate the fraction of the speed of light the electron has.

Here, γ depends on v/c, so after some algebra,

$$\frac{v}{c} = \sqrt{1 - \frac{1}{\left(1 + \frac{K}{m_e c^2}\right)^2}} = 0.139$$

Again, 14% of the speed of light.

(d) Compare your results in (b) and (c). Do you think relativity was important enough to account for in your lab?

Answer: The answers are very similar; you did not need relativity. While very fast, the electrons are not close enough to the speed of light for relativistic effects to become important.

(e) Say you got a lot more expensive equipment that could provide an accelerating voltage of up to $V_a = 500\,\text{kV}$. In that case, what would you calculate the speed of the electrons to be (as a fraction of the speed of light) if you used $K = \frac{1}{2}m_e v^2$?

Answer: Same calculation:

$$500\,\text{keV} = \frac{1}{2}(511\,\text{keV})\frac{v^2}{c^2} \quad \Rightarrow \quad \frac{v}{c} = \sqrt{\frac{2 \cdot 500}{511}} = 1.40$$

140% of the speed of light. That should raise an eyebrow.

(f) Redo the calculation for v as a fraction of the speed of light with $V_a =$ 500 kV, but now using the correct expression for kinetic energy.

Answer: Same calculation:

$$\frac{v}{c} = \sqrt{1 - \frac{1}{\left(1 + \frac{K}{m_e c^2}\right)^2}} = 0.863$$

86% of the speed of light.

(g) Compare your results with different expressions for kinetic energy in (e) and (f) and interpret what they mean.

Answer: Clearly the nonrelativistic calculation is wrong: faster than light? The electrons are now moving close enough to the speed of light that relativity becomes important, and the proper relativistic kinetic energy expression is necessary.

(h) Again, $V_a = 500$ kV. In the lab reference frame, the copper coils with the current providing the magnetic field were circles with a radius of about $R = 6.8$ cm. Sketch how the coils look in the electrons' reference frame, and calculate the appropriate dimensions (height, width) for the coil in that frame.

Answer: Here, $\gamma = 1.98$. The coil is not moving in the lab frame, therefore its dimensions are proper lengths. Length contraction will occur along the direction of motion, so the width will contract down to $2 \cdot 6.8/1.98 = 6.9$ cm. The height is perpendicular to the direction of motion, so this will not be contracted, remaining at $2 \cdot 6.8 = 13.6$ cm. The circle will now look like an ellipse.

3. (30 points) Say you're observing light from a distant star. But the star is moving relative to Earth: it is moving directly away from Earth with speed v. Now, light is an electromagnetic wave. In the Earth frame of reference, the waves have period T_E: the star emits wavefronts once every T_E. Since the star is moving away, on Earth, you won't observe the same time interval T_E between wavefronts.

(a) In the Earth frame, during the time interval T_E it takes between emitting wavefronts, how much further will the star have moved away from the Earth?

Answer: With speed v, the distance covered in time T_E is $d = T_E v$.

(b) How much extra time will the next wavefront take to reach the Earth?

Answer: With an extra distance d and a wave traveling at speed c, the extra time is $d/c = T_E v/c$.

(c) The period you you actually observe, T_o, will be T_E plus the extra time you calculated for the wavefront. What is this observed period?

Answer: $T_o = T_E + T_E v/c = T_E(1 + v/c)$.

(d) Now, remember that T_E was in the Earth frame of reference. What is T_s, the time between wavefront emissions, in the frame of the star (the source)?

Answer: T_s is the proper time, since the wavefront emission events all happen at the same location in the star's frame of reference. Therefore

$$T_E = \gamma T_s = \frac{T_s}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} \quad \text{or} \quad T_s = T_E \sqrt{1 - \left(\frac{v}{c}\right)^2}$$

(e) Put everything together: find an equation for T_o in terms of T_s and v/c. (If you want to simplify your expression, note that $(1 - a^2) = (1 + a)(1 - a)$.)

Answer:

$$T_o = \frac{T_s}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} \left(1 + \frac{v}{c}\right) = T_s \frac{1 + \frac{v}{c}}{\sqrt{\left(1 + \frac{v}{c}\right)\left(1 - \frac{v}{c}\right)}} = T_s \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}}$$

(f) Check your equation. What does it give you when $v = 0$? Is this reasonable? What about when $v = c$? Is that what you'd expect—if the source is moving away from you at $v = c$, would you see any waves?

Answer: When $v = 0$, $T_o = T_s$—since the star and the Earth are in the same reference frame (at rest relative to each other), the periods should be the same. When $v \to c$, $T_o \to \infty$. You don't see any wave, as there is an infinite delay between wavefronts. Well, that's what will happen if the wave source is receding from you at the same speed as the wave.

(g) Let's say that since you know the physics of stars, you know that the star you're observing should be emitting blue light with a wavelength of 450 nm. But what you actually see is red light with a wavelength of 700 nm. What, then, is v/c, the fraction of the speed of light the star is moving away from Earth?

Answer: Since wave speed is λf and $f = 1/T$, the period of the light wave is $T = \lambda/c$. The ratio

$$\frac{T_o}{T_s} = \frac{\lambda_o/c}{\lambda_s/c} = \frac{\lambda_o}{\lambda_s}$$

But that ratio is also in the previous answer:

$$\frac{T_o}{T_s} = \frac{\lambda_o}{\lambda_s} = \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}$$

The wavelength ratio $r = \lambda_o/\lambda_s = 700/450$. We need to solve for v/c. Squaring both sides,

$$r^2 = \frac{1+\frac{v}{c}}{1-\frac{v}{c}} \quad \Rightarrow \quad \frac{v}{c} = \frac{r^2-1}{r^2+1} = 0.415$$

This is the famous *redshift* of light from stars and galaxies receding away from us.

Extra Problems (not graded)

4. (0 points) A cosmic ray collision creates a muon (a subatomic particle) near the top of the troposphere, at an altitude of 9000 m. The muon heads straight towards the surface at a speed of $0.998c$.

(a) In the reference frame of a ground observer, what is the muon's initial distance to the surface? What is the time the muon takes to reach the surface?

Answer: In this reference frame you measure Δx_0, which is 9000 m. The time to the surface is $(9000 \text{ m})/(0.998c) = 3.01 \times 10^{-5}$ s. There is no relativity involves, since we do not switch between frames of reference: the question entirely concerns the ground frame if reference. So no γ factors are involved.

This time interval is Δt, a dilated time. Since in the ground frame, the creation of the muon and its reaching the surface take place at different locations (9000 m high and 0 m high), this is not the proper time. However, since the troposphere and the surface are stationary in the ground frame, the proper length $\Delta x = 9000$ m is in this frame.

(b) In the reference frame of the muon, what is the muon's initial distance to the surface? What is the time the muon takes to reach the surface?

Answer: In the muon's frame of reference, the ground is rushing up to meet the muon: the beginning and ending locations are not stationary. Therefore the length (height) in the muon's frame will be contracted: $\Delta x = \Delta x_0/\gamma$. The time dilation factor $\gamma = 1/\sqrt{1 - 0.998^2} = 15.8$. So $\Delta x = 569$ m. In this reference frame the ground is rushing up at $v = 0.998c$. The time it takes to meet it is $\Delta x/v = 1.9 \times 10^{-6}$ s. This is Δt_0, since the beginning and end events (muon being created and muon reaching the ground) happen at the same location: exactly where the muon is.

(c) When measured at rest in the lab, the average lifetime of a muon is 2.2×10^{-6} s. Given your answers to (a) and (b), would an average muon make it to the surface, or does it have to be an exceptionally long-lived one? Explain.

Answer: The muon's frame of reference is where the muon is stationary, as in the lab experiments that established the average lifetime. Therefore we should compare Δt_0 we just calculated to the lifetime. Since $1.9 \times 10^{-6} < 2.2 \times 10^{-6}$, the average muon has enough time to make it to the surface.

5. (0 points) You have a proton and an antiproton at rest on Earth. They annihilate to produce a muon-antimuon pair: $p + \bar{p} \rightarrow \mu^- + \mu^+$. The muon heads toward the Moon, 3.8×10^8 m away, and the antimuon is captured by a detector here on Earth. The typical lifetime of a muon is 2.2×10^{-6} s. Will the muon make it to the Moon to be captured by a detector there? A muon's mass is $m_\mu = 1.9 \times 10^{-28}$ kg, or 110 MeV/c^2. A proton's mass is $m_p = 1.7 \times 10^{-27}$ kg or 940 MeV/c^2. The speed of light is 3.0×10^8 m/s. Note:

- Relativistic energy (γmc^2) and momentum ($\gamma m\vec{v}$) are both conserved in this reaction. Show how you use both.
- You'll get a **bonus +5 points** if you solve this using the masses given in MeV/c^2.

Answer: The masses of particles and their antiparticles are identical. Therefore, with the proton and antiproton starting from rest, momentum conservation looks like

$$0 = \gamma_{\mu^-} m_\mu \vec{v}_{\mu^-} + \gamma_{\mu^+} m_\mu \vec{v}_{\mu^+}$$

This implies the velocities of the muon and antimuon are equal and opposite, and that therefore $\gamma_{\mu^-} = \gamma_{\mu^+}$.

Energy conservation then gives, with $\gamma_p = \gamma_{\bar{p}} = 1$ because they are at rest,

$$2m_p c^2 = 2\gamma_\mu m_\mu c^2 \qquad \Rightarrow \qquad \gamma_\mu = \frac{m_p}{m_\mu} = 8.6$$

This is a large γ, which implies the speed of the muon $v \approx c$. In the frame of reference of an observer on Earth, the muon's lifetime will increase due to time dilation; it will live for $\Delta t = \gamma_\mu (2.2 \times 10^{-6}\ \text{s})$. During this time it will travel close to the speed of light, covering the proper distance

$$\Delta x_0 = c\gamma_\mu (2.2 \times 10^{-6}\ \text{s}) = 5.6 \times 10^3\ \text{m} < 3.8 \times 10^8\ \text{m}$$

The muon will fall far short of the Moon.