Solutions to Assignment 8; Phys 186

1. (30 points) A physics major tells you that they have a new explanation of black holes. They ask you to imagine a brick, which we heat up to higher and higher temperatures.

- Atoms are made of electrically charged particles.
- As temperature increases, the atoms making up the brick will jiggle more vigorously.
- More vigorously jiggling atoms will radiate higher amplitude electromagnetic waves.
- More electromagnetic waves in the brick will lead to increased destructive interference between the waves;
- Therefore the intensity of the radiation escaping from the brick will decrease.
- Since mass is equivalent to energy, heating up the brick while less and less radiation escapes is equivalent to adding more and more mass.
- Beyond a threshold, *no* radiation will escape: the brick will be completely black. We can calculate the threshold using $E = mc^2$.

This reasoning is incorrect. Circle the bullet points that seem wrong or misleading to you and briefly explain why they make the reasoning incorrect. Make calculations where needed: for example, compare the energy needed to raise the brick's temperature by $1\,\mathrm{K}$ (revisit your first semester physics) to mc^2.

Answer: Things start falling apart with the "increased destructive interference" point. No such thing happens; thermal motion is random jiggling, so the radiation due to thermal motion is a chaotic mess of different phases, amplitudes, and wavelengths that cannot produce any coherent interference of any sort. In fact, there will be an increasing intensity of radiation, proportional to T^4 as we learned in the first semester (and as problem 3 should remind you). The rest is blather; while adding more energy *is* equivalent to adding more mass, the brick will be vaporized long before any measurable increase in mass will take place. That's because the energy that you need to heat up a brick is much much smaller than mc^2 for a brick. Taking the ratio of the energy needed to heat a brick by 1 K to mc^2, using a typical number you can look up for the specific heat of a brick, $c_b = 900\,\mathrm{J/kg{\cdot}K}$, we get

$$\frac{mc_b\Delta T}{mc^2} = \frac{900\,\mathrm{J/kg}}{9\times 10^{16}\,\mathrm{J/kg}} = 10^{-14}$$

That is a ridiculously small number.

2. (30 points) For the following, you will need the expression for the radius of the event horizon of a black hole that we derived.

(a) One way to get a black hole is to squeeze lots of mass into a very small volume. Almost all the mass of a Hydrogen atom is squeezed into the very small volume of a proton. Are protons dense enough to be black holes? Answer this question by using reasonable numbers about protons you can look up. Write down the sources of your numbers.

Answer: The event horizon of a black hole has radius $r = 2Gm/c^2$. For the proton mass of $m = 1.67\times 10^{-27}$ kg, we get $r = 2.48\times 10^{-54}$ m. The radius of the proton is around 10^{-15} m, since that is the order of magnitude for nuclear distances. Whatever the exact value you take, clearly the proton radius is *much, much* larger than what it would need to be for a black hole.

(b) You stand such that your lower legs, with mass 5.0 kg, is at the event horizon of a 30 solar mass black hole, and your 5.0 kg head is just 1.0 m further away from the horizon. Find the difference between the gravitational forces felt by your head and your lower legs. What effect on you would this difference in forces have? (Math hint: When $d \ll r$, $\frac{1}{r^2} - \frac{1}{(r+d)^2} \approx \frac{2d}{r^3}$.)

Answer: The difference between gravitational forces is

$$\Delta F = GMm\left(\frac{1}{r^2} - \frac{1}{(r+d)^2}\right) \approx \frac{2GMmd}{r^3}$$

where $m = 5$ kg, $M = 30\times 1.99\times 10^{30}$ kg, $d = 1$ m, and $r = 2GM/c^2 = 8.85\times 10^4$ m. With all the numbers in, we get $\Delta F = 5.7\times 10^7$ N. This is very large—you will be ripped apart.

3. (40 points) If you incorporate quantum effects, it turns out that black holes have a temperature. And as with any object that has a temperature, black holes therefore emit thermal radiation. If you look up the black hole surface temperature, you will find

$$T = \frac{hc^3}{2^4 \pi^2 G k_B M}$$

Where h is Planck's constant, c is the speed of light, G is the universal gravitational constant, k_B is the Boltzmann constant, and M is the mass of the black hole. You will remember, from your first semester of physics, that the rate of heat loss by electromagnetic radiation from an object at temperature T is $dQ/dt \propto T^4$ (you may have seen this as $Q/\Delta t \propto T^4$; it's the same thing). Go look up that equation, and

(a) Find the equation for the *rate of mass loss* of a black hole due to thermal radiation. Simplify the expression as much as you can.

Answer: The radiative heat loss equation is $dQ/dt = \sigma e A T^4$. Here, σ is a constant; you may also find a version where $\sigma = 2\pi^5 k_B^4 / 15 c^2 h^3$. The emissivity $e = 1$ for a black hole—it's black. A is the surface area; for a black hole, this is $4\pi R^2$ where the event horizon radius is $R = 2GM/c^2$.

The heat loss rate is an energy loss rate. But you know that energy is the same as mass. Since $E = mc^2$, the mass loss rate $dM/dt = \frac{1}{c^2} dQ/dt$. Putting it all together and simplifying,

$$\frac{dM}{dt} = \frac{\sigma c^6 h^4}{2^{12} \pi^7 G^2 M^2 k_B^4} = \frac{hc^4}{15 \cdot 2^{11} \pi^2 G^2 M^2}$$

Note that $dM/dt \propto 1/M^2$, so the larger the black hole, the *smaller* its rate of evaporation.

(b) Calculate the rate of mass loss of a supermassive black hole at the center of a galaxy with $M = 10^9\, M_\odot$, where $M_\odot$ stands for a solar mass. You should find that such a large black hole is not in danger of evaporating—explain why your result means this.

Answer: The numbers you need are $h = 6.63 \times 10^{-34}$ J·s, $c = 3.00 \times 10^8$ m/s, $G = 6.67 \times 10^{-11}$ m^3/kg·s^2, $k_B = 1.38 \times 10^{-23}$ J/K, $\sigma = 5.67 \times$

$10^{-8}\,\mathrm{W/m^2{\cdot}K^4}$, $M_\odot = 1.99 \times 10^{30}\,\mathrm{kg}$. Putting these into the above equation,

$$\frac{dM}{dt} = 10^{-63}\,\mathrm{kg/s}$$

This is an absurdly small rate of mass loss, especially compared to such a large mass for the black hole itself. Large black holes are permanent.

(c) Find the mass of a microscopic black hole where the rate of mass loss is equal to its own mass per second. If you could somehow compress an everyday object into a black hole, would it last?

Answer: Set $dM/dt = M/(1\,\mathrm{s})$ and solve for the mass:

$$M = \left(\frac{hc^4(1\,\mathrm{s})}{15 \cdot 2^{11}\pi^2 G^2}\right)^{1/3} = 1.58 \times 10^5\,\mathrm{kg}$$

So a black hole of about 150 tons will vanish in about a second, with an explosion that will make nuclear warfare look like a mild fistfight. If you could compress everyday objects into black holes, they would *not* last; and they would be the most destructive things we have ever invented. Fortunately, there's no known way to make such black holes, so we will have to rely on ordinary nuclear weapons and global warming to destroy civilization.

Extra Problems (not graded)

4. (0 points) One estimate for the average mass density of our universe is the mass of one H atom per 3 cubic meters.

(a) What is the radius of a black hole that has that mass density? Compare your result to the size of our *observable* universe (look it up).

Answer: The mass density is

$$\rho = \frac{m_H}{V} = \frac{1.67 \times 10^{-27}}{3} = 5.67 \times 10^{-28}\,\mathrm{kg/m^3}$$

Now for a black hole,

$$r = \frac{2GM}{c^2} \quad \Rightarrow \quad M = \frac{c^2 r}{2G} \qquad \text{and} \qquad \rho = \frac{M}{\frac{4}{3}\pi r^3} = \frac{c^2}{\frac{8}{3}\pi G r^2}$$

Solving for r,

$$r = \frac{c}{\sqrt{\frac{8}{3}\pi G\rho}} = 5.33 \times 10^{26}\text{ m}$$

If you look up the radius of the observable universe, you will find a number around 4.4×10^{26} m. These numbers are very similar; we might be living inside a black hole!

(b) A friend of yours now suggests that it is impossible that we live inside a black hole. After all, everything that falls into a black hole gets ripped apart and gets compressed into an incredibly high density. But we're alive! What do you think of that argument? Explain.

Answer: Black holes with different sizes can behave very differently. Since the black hole radius $r \propto M$ and the volume $V \propto r^3$, this means that the density of a black hole $\rho \propto 1/r^2$: it *falls* rapidly as the black hole size increases. So nothing inside needs to be compressed to a high density, provided the black hole is universe-sized.

5. (0 points) You stand such that your lower legs, with mass 5.0 kg, is at the event horizon of a black hole with mass M, and your 5.0 kg head is just 1.0 m further away from the event horizon.

(a) Find the difference ΔF between the gravitational forces felt by your head and your lower legs, for black holes with mass $M = 10^{18}$ kg, 10^{21} kg, 10^{24} kg, 10^{27} kg, 10^{30} kg, 10^{33} kg, 10^{36} kg, and 10^{39} kg.

When $d \ll r$, $\left(\frac{1}{r^2} - \frac{1}{(r+d)^2}\right) \approx \frac{2d}{r^3}$. When $d \gg r$, $\left(\frac{1}{r^2} - \frac{1}{(r+d)^2}\right) \approx \frac{1}{r^2}$. If, however, r and d are comparable in magnitude, you will have to calculate ΔF without using either of these approximations.

Answer: The radius of each black hole is $2GM/c^2$, which is $2G/c^2 = 1.48 \times 10^{-27}$ m/kg times the mass M. Therefore, the $M = 10^{27}$ kg black hole will require a full calculation for ΔF, while the others can be handled by the approximations.

For $M < 10^{27}$ kg, the $d \gg r$ approximation will work:

$$\Delta F = GMm\left(\frac{1}{r^2} - \frac{1}{(r+d)^2}\right) \approx \frac{GMm}{r^2} = \frac{mc^4}{4GM} = \frac{1.52 \times 10^{44}\,\mathrm{N}\cdot\mathrm{kg}}{M}$$

$$M = 10^{18}\,\mathrm{kg} \quad \Rightarrow \quad \Delta F = 1.52 \times 10^{26}\,\mathrm{N}$$
$$M = 10^{21}\,\mathrm{kg} \quad \Rightarrow \quad \Delta F = 1.52 \times 10^{23}\,\mathrm{N}$$
$$M = 10^{24}\,\mathrm{kg} \quad \Rightarrow \quad \Delta F = 1.52 \times 10^{20}\,\mathrm{N}$$

With $M = 10^{27}$ kg,

$$\Delta F = GMm\left(\frac{1}{r^2} - \frac{1}{(r+d)^2}\right) = 9.8 \times 10^{16}\,\mathrm{N}$$

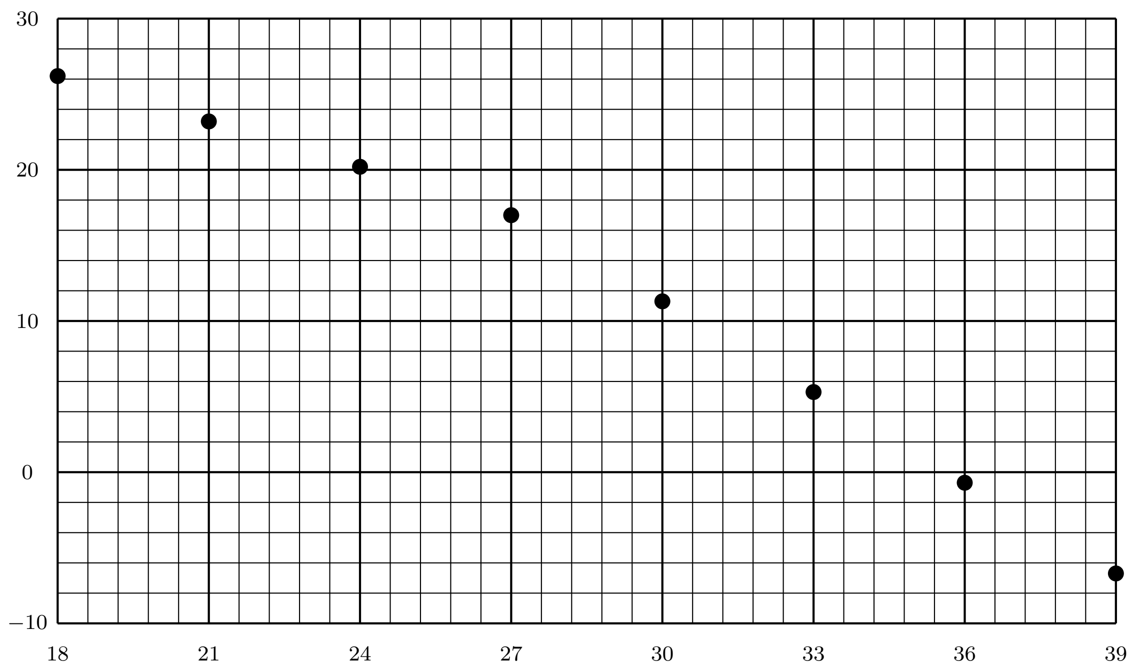
For $M > 10^{27}$ kg, the $d \ll r$ approximation will work:

$$\Delta F = GMm\left(\frac{1}{r^2} - \frac{1}{(r+d)^2}\right) \approx \frac{2GMmd}{r^3} = \frac{mdc^6}{4G^2M^2} = \frac{2.05 \times 10^{71}\,\mathrm{N}\cdot\mathrm{kg}^2}{M^2}$$

$$M = 10^{30}\,\mathrm{kg} \quad \Rightarrow \quad \Delta F = 2.05 \times 10^{11}\,\mathrm{N}$$
$$M = 10^{33}\,\mathrm{kg} \quad \Rightarrow \quad \Delta F = 2.05 \times 10^{5}\,\mathrm{N}$$
$$M = 10^{36}\,\mathrm{kg} \quad \Rightarrow \quad \Delta F = 2.05 \times 10^{-1}\,\mathrm{N}$$
$$M = 10^{39}\,\mathrm{kg} \quad \Rightarrow \quad \Delta F = 2.05 \times 10^{-7}\,\mathrm{N}$$

(b) Make a graph of $\log_{10}\Delta F$ on the vertical axis and $\log_{10} M$ on the horizontal axis. Connect the dots. On the graph, also indicate: the mass of the Earth, the mass of the sun, the mass of a 30 solar mass black hole, and the mass of a thousand solar mass black hole. Then, also indicate where the tidal stretching will begin to be dangerous for humans: if a tension force of 1000 N were applied to us, we could only withstand that with great pain after just minutes.

Note that it's important in science to communicate your results effectively. I expect you to make good choices and a clear presentation with this graph.



If a spaceship were to approach the event horizon of a supermassive black hole with a billion solar masses, would they have to worry about tidal forces?

Answer: No. You can clearly see from the graph that the tidal force ΔF would be tiny in this case.