Homework Solutions 3 (Griffiths Chapter 3)

13 Call (a, b) the location $\mathbf{r}_1$. Put image charges $-q$ at $(-a, b), (a, b)$ (locations $\mathbf{r}_2$ and $\mathbf{r}_3$) and q at $(-a, -b)$ (location $\mathbf{r}_4$). The potential is then

$$V(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{|\mathbf{r}-\mathbf{r}_1|} - \frac{1}{|\mathbf{r}-\mathbf{r}_2|} - \frac{1}{|\mathbf{r}-\mathbf{r}_3|} + \frac{1}{|\mathbf{r}-\mathbf{r}_4|}\right)$$

The force on the real charge is,

$$\mathbf{F}(\mathbf{r}) = \frac{q^2}{4\pi\epsilon_0}\left(-\frac{1}{4b^2}\hat{\mathbf{y}} - \frac{1}{4a^2}\hat{\mathbf{x}} + \frac{1}{[4(a^2+b^2)]^{3/2}}(2a\hat{\mathbf{x}} + 2b\hat{\mathbf{y}})\right)$$

The work is, bringing the charge in from infinity along the $y = b$ line,

$$\begin{aligned} W &= -\frac{q^2}{4\pi\epsilon_0}\int_a^\infty dx\left(-\frac{x-a}{[4b^2+(x-a)^2]^{3/2}} - \frac{1}{(x+a)^2}\right. \\ &\qquad \left. + \frac{x+a}{[4b^2+(x+a)^2]^{3/2}}\right) \\ &= \frac{q^2}{4\pi\epsilon_0}\left(\frac{1}{2b} + \frac{1}{2a} - \frac{1}{2\sqrt{a^2+b^2}}\right) \end{aligned}$$

If we were able to arrange the charge and image charges into a hexagon, octagon, and so forth, with alternating signs for the charges, we would also get $V = 0$ along planes. So any angle of $360°/2n = \pi/n$ would work, for $n = 1, 2, 3, \ldots$

17 It's most convenient to use the hyperbolic solutions for basis functions along the x-axis:

$$V_k(x, y) = (A_k \sinh kx + B_k \cosh kx)(C_k \sin ky + D_k \cos ky)$$

Putting in the boundary conditions at $x = 0$ and $y = 0$, this means $B_k = 0$ and $D_k = 0$. Having $V = 0$ for $y = a$ requires $k = n\pi/a$. The general solution giving $V = V_0(y)$ at $x = b$ is therefore

$$V(x, y) = \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi}{a}x\right)\sin\left(\frac{n\pi}{a}y\right)$$

Since

$$V_0(y) = \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi}{a}b\right) \sin\left(\frac{n\pi}{a}y\right)$$

this means that A_n can be found using the orthogonality relationship between sine functions. Operating on each side with $\frac{2}{a}\int_0^a dy\, \sin(\frac{m\pi}{a}y)$ results in

$$A_m = \frac{2}{a\sinh\left(\frac{m\pi}{a}b\right)} \int_0^a dy\, \sin\left(\frac{m\pi}{a}y\right) V_0(y)$$

When $V_0(y) = V_0$,

$$A_n = \frac{4V_0}{\pi \sinh\left(\frac{n\pi}{a}b\right)} \begin{cases} 0 & n \text{ even} \\ \frac{1}{n} & n \text{ odd} \end{cases}$$

21 Since the voltage should be finite as $r \to 0$ and $r \to \infty$, the solutions are

$$V_{\text{in}} = A_l\, r^l\, P_l(\cos\theta) \qquad V_{\text{out}} = B_l\, r^{-(l+1)}\, P_l(\cos\theta)$$

At $r = R$, we have, using the orthogonality of P_l functions,

$$A_l = \frac{2l+1}{2R^l} \int_0^{\pi} d\theta\, \sin\theta\, P_l(\cos\theta)\, k\cos 3\theta$$

To do these integrals, notice that

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta = \frac{8}{5}P_3(\cos\theta) - \frac{3}{5}P_1(\cos\theta)$$

Therefore

$$A_l = \frac{k(2l+1)}{2R^l} \int_{-1}^{1} dx\, P_l(x) \left[\frac{8}{5}P_3(x) - \frac{3}{5}P_1(x)\right]$$

This integral will be non-zero only for $l = 1$ and $l = 3$. We get

$$A_1 = -\frac{3k}{5R} \qquad A_3 = \frac{8k}{5R^3}$$

The continuity of the potential means $A_l\, R^{2l+1} = B_l$. The continuity of the normal derivatives gives

$$\begin{aligned} \sigma &= -\epsilon_0 \left[\frac{\partial V_{\text{out}}}{\partial r} - \frac{\partial V_{\text{in}}}{\partial r}\right]_{r=R} = -\epsilon_0 \left[B_1(-2)R^{-3}P_1(\cos\theta)\right. \\ &\qquad \left. + B_3(-4)R^{-5}P_3(\cos\theta) - A_1(1)P_1(\cos\theta) - A_3(3)P_3(\cos\theta)\right] \\ &= \frac{k\epsilon_0}{R}\left[\frac{56}{5}P_3(\cos\theta) - \frac{9}{5}P_1(\cos\theta)\right] = \frac{k\epsilon_0}{R}(28\cos^3\theta - 15\cos\theta) \end{aligned}$$

27 Using the cylindrical basis functions,

$$V = A_0 \ln s + B_0 + \sum_{m=1}^{\infty} (A_m s^m + B_m s^{-m})(C_m \cos m\phi + D_m \sin m\phi)$$

As $s \to \infty$, $V \to -E_0 x = -E_0 s \cos\phi$. Therefore, $A_m = 0$ for $m \neq 1$, and $A_1 = -E_0$. Also, $B_0 = 0$ and $D_1 = 0$. Since $V = 0$ at $s = R$, $B_m = 0$ for $m \neq 0$ as well: we get $A_1 R \cos\phi + B_1 R^{-1} \cos\phi = 0$. Therefore $B_1 = -A_1 R^2 = E_0 R^2$ and

$$V = -E_0\, s \cos\phi \left(1 - \frac{R^2}{s^2}\right)$$

The surface charge again comes through the normal derivatives:

$$\sigma = -\epsilon_0 \left[\frac{\partial V}{\partial s}\right]_{s=R} = 2\epsilon_0 E_0 \cos\phi$$

57 The quadrupole...

(a) With $P_2(x) = \frac{1}{2}(3x^2 - 1)$ and $\cos\alpha = \hat{\mathbf{r}} \cdot \hat{\mathbf{r}}'$, the quadrupole term is

$$V_{\text{quad}} = \frac{1}{4\pi\epsilon_0} \frac{1}{r^3} \int dv' \, \frac{1}{2} \left[3(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}')^2 - 1\right] \, r'^2 \, \rho(\mathbf{r}')$$

Note, now, that

$$(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}')^2 = \left(\sum_i \hat{r}_i \hat{r}'_i\right)^2 = \sum_{ij} \hat{r}_i \hat{r}'_i \hat{r}_j \hat{r}'_j = \sum_{ij} (\hat{r}_i \hat{r}_j)(\hat{r}'_i \hat{r}'_j) = \sum_{ij} (\hat{r}_i \hat{r}_j) \frac{(r'_i r'_j)}{r'^2}$$

$$1 = \hat{\mathbf{r}} \cdot \hat{\mathbf{r}} = \sum_i \hat{r}_i \hat{r}_i = \sum_{ij} \hat{r}_i \hat{r}_j \delta_{ij}$$

Putting these in the integral,

$$V_{\text{quad}} = \frac{1}{4\pi\epsilon_0} \frac{1}{r^3} \sum_{ij} \hat{r}_i \hat{r}_j \int dv' \, \frac{1}{2} \left[3r'_i r'_j - r'^2 \delta_{ij}\right] \, \rho(\mathbf{r}')$$

(b) In this case,

$$\rho(\mathbf{r}) = q\,\delta(z)\Big[\delta(x-\frac{a}{2})\delta(y-\frac{a}{2}) + \delta(x+\frac{a}{2})\delta(y+\frac{a}{2}) - \delta(x-\frac{a}{2})\delta(y+\frac{a}{2}) - \delta(x+\frac{a}{2})\delta(y-\frac{a}{2})\Big]$$

This means that all z-components—all Q_{ij} where $i = 3$ or $j = 3$—are zero. All diagonal components with $i = j$ also turn out to be zero, because of the opposite sign charges. Also notice that the quadrupole moment is symmetric: $Q_{ij} = Q_{ji}$. So we're left with the non-zero components

$$Q_{12} = Q_{21} = \int dv'\,\frac{3}{2}\,x'y'\rho(\mathbf{r}') = \frac{3}{2}\,qa^2$$

(c) When we shift the origin, $\mathbf{r}' \to \mathbf{r}' - \mathbf{a}$. Therefore, $r'_i \to r'_i - a_i$ and

$$r'_i r'_j \to r'_i r'_j - a_i r'_j - a_j r'_i + a_i a_j \quad r'^2 \to \sum_i (r'_i - a_i)^2 = \sum_i (r'^2_i r - 2a_i r'_i + a_i^2)$$

Notice that all the terms added to the integral are either constants or linear in r'_i. The constant terms give contributions proportional to

$$\int dv'\,\rho(\mathbf{r}') = Q = 0$$

The linear terms give contributions proportional to

$$\int dv'\,r'_i\,\rho(\mathbf{r}') = p_i = 0$$

Therefore the result is independent of the origin.

(d) Now, the octopole. We need $P_3(x) = \frac{1}{2}(5x^3 - 3x)$. Then,

$$(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}')^3 = \left(\sum_i \hat{r}_i \hat{r}'_i\right)^3 = \sum_{ijk} (\hat{r}_i \hat{r}_j \hat{r}_k)(\hat{r}'_i \hat{r}'_j \hat{r}'_k) = \sum_{ijk} (\hat{r}_i \hat{r}_j \hat{r}_k)\frac{(r'_i r'_j r'_k)}{r'^3}$$

$$\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}' = \sum_i \hat{r}_i \hat{r}'_i = \sum_i \hat{r}_i \hat{r}'_i \left(\sum_{jk} \hat{r}_j \hat{r}_k \delta_{jk}\right) = \sum_{ijk} (\hat{r}_i \hat{r}_j \hat{r}_k)\frac{(r'_i \delta_{jk})}{r'}$$

We then define

$$V_{\rm oct} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^4}\sum_{ijk}\hat{r}_i\hat{r}_j\hat{r}_k\, O_{ijk}$$

where

$$O_{ijk} = \int dv'\, \frac{1}{2}\left[5r'_ir'_jr'_k - 3r'^2r'_i\delta_{jk}\right]\,\rho(\mathbf{r}')$$