1. (100 points) You have a circular wire with radius R, which carries a constant current I. Now, a current loop is a classic dipole, and the magnetic field due to its dipole moment dominates at even moderate distances from the loop. But such a loop is not an ideal dipole, and so there are also contributions due to higher moments. Find the first non-zero contribution beyond the dipole to the magnetic field produced by the circular wire.

Answer: For the wire, $d\mathbf{l} = R\,d\phi'\,\hat{\boldsymbol{\phi}}'$, and $r' = R$. We also need $\cos\alpha = \hat{\mathbf{r}}'\cdot\hat{\mathbf{r}}$, which requires

$$\hat{\mathbf{r}}' = \hat{\mathbf{s}}' \quad \text{and} \quad \hat{\mathbf{r}} = \frac{s\,\hat{\mathbf{s}} + z\,\hat{\mathbf{z}}}{(s^2+z^2)^{1/2}}$$

Now, $\hat{\mathbf{s}}'\cdot\hat{\mathbf{z}} = 0$, and

$$\hat{\mathbf{s}}'\cdot\hat{\mathbf{s}} = \cos\phi'\cos\phi + \sin\phi'\sin\phi = \cos(\phi'-\phi)$$

Thefore, the quadrupole vector potential for the circular wire is

$$\begin{aligned}
\mathbf{A}_q &= \frac{\mu_0 I}{4\pi}\frac{1}{r^3}\oint_0^{2\pi} d\mathbf{l}\,(r')^2\,P_2(\cos\alpha) \\
&= \frac{\mu_0 I}{4\pi}\frac{1}{r^3}\oint_0^{2\pi} d\phi'\,(R\hat{\boldsymbol{\phi}}')R^2\left[\frac{3}{2}\left(\frac{s}{(s^2+z^2)^{1/2}}\right)^2\cos^2(\phi'-\phi)-\frac{1}{2}\right] \\
&= \frac{\mu_0 I}{8\pi}\frac{R^3}{(s^2+z^2)^{3/2}}\oint_0^{2\pi} d\phi'\,\left(-\sin\phi'\,\hat{\mathbf{x}}+\cos\phi'\,\hat{\mathbf{y}}\right) \\
&\qquad\qquad \left[3\left(\frac{s^2}{s^2+z^2}\right)\cos^2(\phi'-\phi)-1\right] \\
&= 0
\end{aligned}$$

Note that $\hat{\boldsymbol{\phi}}' = -\sin\phi'\,\hat{\mathbf{x}} + \cos\phi'\,\hat{\mathbf{y}}$ was useful in the integral because $\hat{\boldsymbol{\phi}}'$ is not a constant, and the resulting integrals are zero because they're integrals of odd functions over a full period.

In any case, the quadrupole contribution is zero, so we move onto the octopole. Since $P_3(x)$ is an odd function, we now expect nonzero integrals.

$$\begin{aligned}
\mathbf{A}_o &= \frac{\mu_0 I}{4\pi}\frac{1}{r^4}\oint_0^{2\pi} d\mathbf{l}\,(r')^3\,P_3(\cos\alpha) \\
&= \frac{\mu_0 I}{8\pi}\frac{R^4}{(s^2+z^2)^2}\oint_0^{2\pi} d\phi'\,\left(-\sin\phi'\,\hat{\mathbf{x}}+\cos\phi'\,\hat{\mathbf{y}}\right) \\
&\qquad \left[5\left(\frac{s}{(s^2+z^2)^{1/2}}\right)^3\cos^3(\phi'-\phi)-3\left(\frac{s}{(s^2+z^2)^{1/2}}\right)\cos(\phi'-\phi)\right]
\end{aligned}$$

Now, use these integral results:

$$\oint_0^{2\pi} d\phi' \, \sin\phi' \cos^3(\phi' - \phi) = \frac{3}{4}\pi \sin\phi \qquad \oint_0^{2\pi} d\phi' \, \sin\phi' \cos(\phi' - \phi) = \pi \sin\phi$$

$$\oint_0^{2\pi} d\phi' \, \cos\phi' \cos^3(\phi' - \phi) = \frac{3}{4}\pi \cos\phi \qquad \oint_0^{2\pi} d\phi' \, \cos\phi' \cos(\phi' - \phi) = \pi \cos\phi$$

Then,

$$\begin{aligned}
\mathbf{A}_o &= \frac{3\mu_0 I}{8}\frac{R^4}{(s^2+z^2)^2}\left[\frac{5}{4}\left(\frac{s^3}{(s^2+z^2)^{3/2}}\right) - \left(\frac{s}{(s^2+z^2)^{1/2}}\right)\right] \\
&\qquad (-\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}) \\
&= \frac{3\mu_0 I}{8}\frac{R^4}{(s^2+z^2)^2}\left[\frac{5}{4}\left(\frac{s^3}{(s^2+z^2)^{3/2}}\right) - \left(\frac{s}{(s^2+z^2)^{1/2}}\right)\right]\hat{\boldsymbol{\phi}}
\end{aligned}$$

The magnetic field due to this vector potential is

$$\begin{aligned}
\mathbf{B}_o &= \boldsymbol{\nabla} \times \mathbf{A}_o \\
&= \frac{3\mu_0 I}{8}\frac{R^4}{(s^2+z^2)^{5/2}}\left[\frac{zs}{s^2+z^2}\left(\frac{35}{4}\frac{s^2}{s^2+z^2} - 5\right)\hat{\mathbf{s}} - \right. \\
&\qquad \left.\left(\frac{35}{4}\frac{s^4}{(s^2+z^2)^2} - 10\frac{s^2}{s^2+z^2} + 2\right)\hat{\mathbf{z}}\right]
\end{aligned}$$