Solutions to Exam 3; Phys 186

1. (25 points) Very recently, signs of possible life have been observed on the nearby planet K2-18b, which is 124 light years away. Imagine that we decided to investigate by sending a human mission to K2-18b. The rocket ship crew consists of two astronauts with masses of 50 kg each. (For astronauts, lighter is better.) We don't want the crew to die of old age, so let's say they travel at a constant speed such that they age only 2 years during the journey. Ignore all the complications associated with the crushing accelerations they will have to endure to reach such a high speed. And ignore the life support systems, equipment, rocket, and fuel needed for their trip.

(a) Calculate the energy needed to accelerate the crew (only the crew) to such a speed.

Answer: Aging only 2 years requires a very high γ, which means $v \approx c$. So the dilated time of travel, in the Earth frame, must be just about 124 years. In that case, $\gamma = 124/2 = 62$.

Accelerating the crew up to such a speed requires kinetic energy: $K = (\gamma - 1)mc^2 = 5.49 \times 10^{20}$ J.

(b) Currently, the energy production of the USA is about 10^{20} J per year. Assume all of that were used to accelerating the crew. How long would it take to accumulate all the energy needed?

Answer: $5.49 \times 10^{20}/10^{20} = 5.49$ years.

(c) Given your calculations, and the fact that getting the crew there would be only a tiny fraction of the energy cost, what do you think about the prospects for human travel to even the closest stars in the near future?

Answer: There is no way we can produce such absurdly high energies and devote it to space travel.

2. (25 points) A major recent accomplishment in physics is detecting gravitational waves from events such as the collision and merger of black holes. Let's say you merge two black holes, each with a mass of $M = 25\, M_\odot$, where $M_\odot$ stands for a solar mass.

(a) Calculate the total volume enclosed within the two black holes' event horizons before they merged. (You can either produce an equation involving M or give a numerical result.)

Answer: For two black holes,

$$V = 2\frac{4}{3}\pi\left(\frac{2GM}{c^2}\right)^3 = \frac{2^6}{3}\pi\frac{G^3M^3}{c^6} = 3.37 \times 10^{15}\,\mathrm{m}^3$$

Not all that large.

(b) Calculate the volume enclosed within the event horizons of the merged black hole. (You can either produce an equation involving M or give a numerical result.)

Answer: Merging two identical black holes produces a single black hole with double the mass.

$$V = \frac{4}{3}\pi\left(\frac{2G2M}{c^2}\right)^3 = \frac{2^8}{3}\pi\frac{G^3M^3}{c^6} = 1.35 \times 10^{16}\,\mathrm{m}^3$$

This is four times the previous answer.

(c) Interpret your result. If you were to merge two glasses of water with mass M and volume V, you'd get a large glass of water with mass $2M$ and volume $2V$. How are black holes different?

Answer: Black hole densities vary with the mass, unlike a glass of water. Merging tow black holes produces a black hole with a larger volume than the sum of the opriginal black hole volumes.

3. (30 points) Here is an argument for why the charge on a discharging capacitor shows an exponential decay.

Once the switch is closed, the loop equation gives $V_C + V_R = 0$, which means $\frac{1}{C}Q = -RI$. The current is due to the flow of charges from the capacitor, therefore $I = \frac{d}{dt}Q$. We end up with a differential equation:

$$\frac{d}{dt}Q = -\frac{1}{\tau}Q \qquad \text{with} \qquad \tau = RC$$

Since the rate of change of an exponential is $\frac{d}{dt}e^{-t/\tau} = -\frac{1}{\tau}e^{-t/\tau}$, this means that the solution to our differential equation is

$$Q = Q_0\, e^{-t/\tau} \qquad \text{with} \qquad \tau = RC$$

The equation we've been using for radioactive decays, $m = m_0\, e^{-t/\tau}$, is also an exponential decay. (Recall Assignment 8.) The argument for why it's an exponential decay is mathematically almost identical. But the physical details are different. Construct an argument showing that radioactive decays are described by an exponential decay. Don't forget to address the following points:

- The key is to establish that the rate of change of the total number of radioactive atoms is proportional to the number of radioactive atoms. So you need to consider the meaning of the half-life $t_{1/2}$.
- A *loss* of radioactive atoms (because they decay) means a negative rate of change.
- The total number of decays during a half-life is an expectation value (an average). A measurable mass m means that you're dealing with a huge population of individual atoms.

Answer: Each radioactive nucleus has a fixed probability of decaying in a particular interval of time. And each decay event is *independent*. Therefore, since the rate of change of the number of atomic nuclei is just the change in number of nuclei per unit time, the bigger the number of nuclei, the bigger the rate of change will be. One detail: since we're dealing with probabilities, we can predict *averages* rather than exact values:

$$\frac{d}{dt}N_{\text{avg}} \propto N_{\text{avg}} \qquad \text{or} \qquad \frac{d}{dt}N_{\text{avg}} = -\frac{1}{\tau}N_{\text{avg}}$$

Here, we can call the proportionality constant the inverse of a time scale τ, and there is a minus sign because the rate of change is a *loss* of nuclei.

With macroscopic populations of nuclei, expectation values (averages) become almost certain predictions, as the relative size of fluctuations become negligible. The mass of each radioactive nucleus is identical. Therefore,

$$N_{\text{avg}} = N_0 e^{-t/\tau} \qquad \Rightarrow \qquad m = m_0\, e^{-t/\tau}$$

4. (20 points) The energy levels of a H atom are $E_n = -|E_1|/n^2$, where $n = 1, 2, 3, \ldots$ and $|E_1| = 13.6\,\text{eV}$. An electron in H makes a transition from the $n = 3$ energy level to the $n = 1$ energy level, resulting in the emission of a photon. What would the effect be of spending time in close proximity with a noticeably intense source of such photons?

(a) The photons would barely interact with our molecules, so we wouldn't notice anything. (Radio waves would be an example of this.)

(b) You'd heat up, but otherwise there would be no consequences. (Infrared to red photons would do this.)

(c) You'd worry about both heating and possibly some damage to biological molecules in your skin. (Blue to ultraviolet photons.)

(d) You'd worry mainly about the radiation ripping apart molecules, including in the interior of your body. (X-ray to Gamma radiation.)

Explain your choice.

Answer: Energy conservation means the energy of the emitted photon is equal to the difference between energy levels:

$$E = hf = E_3 - E_1 = (-13.6\,\text{eV})\left(\frac{1}{3^2} - \frac{1}{1^2}\right) = 12.1\,\text{eV}$$

You can look up where such a photon is on the electromagnetic spectrum, or it may be easier to find the wavelength first.

$$\lambda = \frac{hc}{E} = 1.03 \times 10^{-7}\,\text{m}$$

A 103 nm photon is an ultraviolet photon.