Homework Solutions 4 (Griffiths Chapter 4)

10 Using $\mathbf{P} = kr\hat{\mathbf{r}}$, we have

$$\sigma_b = \mathbf{P}\cdot\hat{\mathbf{r}} = kR \qquad \text{and} \qquad \rho_b = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 kr\right) = -3k$$

Note that both of these are constants, and therefore we have a spherically symmetric charge distribution where Gauss's Law should be easy to use. Inside the sphere, with $r < R$,

$$4\pi r^2 E = \frac{4\pi r^3(-3k)}{3\epsilon_0} \quad \Rightarrow \quad \mathbf{E} = -\frac{kr}{\epsilon_0}\hat{\mathbf{r}}$$

Outside, with $r \geq R$,

$$4\pi r^2 E = \frac{4\pi R^3(-3k)}{3\epsilon_0} + \frac{4\pi R^2(kR)}{\epsilon_0} \quad \Rightarrow \quad \mathbf{E} = 0$$

18 This should be straightforward:

(a) Using Gaussian pillboxes with $\oint d\mathbf{a}\cdot\mathbf{D} = Q_{f,in}$, for both dielectric slabs we get $\mathbf{D} = -\sigma\hat{\mathbf{z}}$.

(b) With $\mathbf{D} = \epsilon\mathbf{E}$,

$$\mathbf{E}_1 = -\frac{\sigma}{2\epsilon_0}\hat{\mathbf{z}} \qquad \text{and} \qquad \mathbf{E}_2 = -\frac{2\sigma}{3\epsilon_0}\hat{\mathbf{z}}$$

(c) Since the dielectric constant is $1 + \chi_e$ and $\mathbf{P} = \epsilon_0\chi_e\mathbf{E}$,

$$\mathbf{P}_1 = \epsilon_0(2-1)\mathbf{E}_1 = -\frac{\sigma}{2}\hat{\mathbf{z}} \qquad \text{and} \qquad \mathbf{P}_2 = \epsilon_0(1.5-1)\mathbf{E}_2 = -\frac{\sigma}{3}\hat{\mathbf{z}}$$

(d) With the standard $\int d\mathbf{l}\cdot\mathbf{E}$ line integral between the plates, the potential difference is

$$V = \frac{\sigma a}{\epsilon_0}\left(\frac{1}{2} + \frac{2}{3}\right) = \frac{7\sigma a}{6\epsilon_0}$$

(e) $\rho_b = -\nabla \cdot \mathbf{P} = 0$ throughout. Between the top plate and the first slab,

$$\sigma_b = \mathbf{P}_1 \cdot \hat{\mathbf{z}} = -\frac{\sigma}{2}$$

Between the bottom plate and the second slab,

$$\sigma_b = -\mathbf{P}_2 \cdot \hat{\mathbf{z}} = \frac{\sigma}{3}$$

In the very middle, these are combined:

$$\sigma_b = \frac{\sigma}{2} - \frac{\sigma}{3} = \frac{\sigma}{6}$$

(f) Using $\Delta E_\perp = \sigma/\epsilon_0$,

$$0 - E_{1z} = \frac{1}{\epsilon_0}\left(\sigma - \frac{\sigma}{2}\right) \quad \Rightarrow \quad \mathbf{E}_1 = -\frac{\sigma}{2\epsilon_0}\hat{\mathbf{z}}$$

$$E_{1z} - E_{2z} = \frac{\sigma}{6\epsilon_0} \quad \Rightarrow \quad \mathbf{E}_2 = -\frac{2\sigma}{3\epsilon_0}\hat{\mathbf{z}}$$