Homework Solutions 7 (Griffiths Chapter 7)

36 The capacitor...

(a) The capacitor charge will be $Q = It$. With a parallel-plate capacitor, $E = Q/Cw = It/\epsilon_0 \pi a^2$:

$$\mathbf{E} = \frac{It}{\epsilon_0 \pi a^2}\,\hat{\mathbf{z}}$$

(b) With the loop between the plates,

$$\mathbf{J}_d = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \frac{I}{\pi a^2}\,\hat{\mathbf{z}}$$

With $\mathbf{J} = 0$, and for $s < a$,

$$\oint d\mathbf{l} \cdot \mathbf{B} = (2\pi s)B = \mu_0 \frac{I}{\pi a^2}(\pi s^2) \quad \Rightarrow \quad \mathbf{B} = \frac{\mu_0 I s}{2\pi a^2}\,\hat{\boldsymbol{\phi}}$$

(c) With this surface, I_d does not contribute. There is an incoming current I from the wire, and an outgoing current I_o on the capacitor plate. Since σ must remain spatially constant, but $Q = \pi a^2 \sigma = It$, the charge density on the part of the plate with radius s must be

$$\sigma = \frac{(I - I_o)t}{\pi s^2}$$

Note also that $I_o = 0$ when $s = a$ (there is nowhere else to go) and $I_o = I$ when $s = 0$. All this means that

$$I_o = \left(1 - \frac{s^2}{a^2}\right)$$

The total current going through the surface is $I - I_o = I\, s^2/a^2$. Therefore,

$$\oint d\mathbf{l} \cdot \mathbf{B} = (2\pi s)B = \mu_0 I \frac{s^2}{a^2} \quad \Rightarrow \quad \mathbf{B} = \frac{\mu_0 I s}{2\pi a^2}\,\hat{\boldsymbol{\phi}}$$

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(a) With the changing magnetic flux, the induced voltage in the circuit (with circular surface S) will be

$$\oint_{\partial S} d\mathbf{l}\cdot\mathbf{E} = -\frac{d}{dt}\int_S d\mathbf{a}\cdot\mathbf{B} = -\alpha\pi r^2$$

Therefore the current is $I = \alpha\pi r^2/R$. (Clockwise if $\mathbf{B}$ is pointing upward from the page and increasing in magnitude.)

(b) To get the voltage difference, we need to get $\mathbf{E}$ in the region, and use $\int d\mathbf{l}\cdot\mathbf{E}$ on the path connecting points P and Q. Note that the integral will be path-dependent, since $\boldsymbol{\nabla}\times\mathbf{E}\neq 0$—our usual intuitions about current, voltage, and resistance can be misleading.

The situation is analogous to Ampère's Law with a long current-carrying wire, but now with the electric field. The symmetry of the situation means that $\mathbf{E}$ will circulate in the $-\hat{\boldsymbol{\phi}}$ direction. Take a loop with radius s:

$$\oint_{\partial S} d\mathbf{l}\cdot\mathbf{E} = E(2\pi s) = -\alpha\pi s^2 \quad\Rightarrow\quad \mathbf{E} = -\frac{\alpha s}{2}\,\hat{\boldsymbol{\phi}}$$

It's better to write this in cartesian coordinates to do the P to Q line integral, as $d\mathbf{l} = dx\,\hat{\mathbf{x}}$ where y is constant and $y = r/\sqrt{2}$.

$$\mathbf{E} = -\frac{\alpha s}{2}\frac{1}{s}\left(-y\hat{\mathbf{x}} + x\hat{\mathbf{y}}\right) \quad\Rightarrow\quad d\mathbf{l}\cdot\mathbf{E} = dx\,\frac{\alpha y}{2} = dx\,\frac{\alpha r}{2\sqrt{2}}$$

The voltage, finally, is

$$-\int_P^Q d\mathbf{l}\cdot\mathbf{E} = -\frac{\alpha r}{2\sqrt{2}}\int_{-r/\sqrt{2}}^{r/\sqrt{2}} dx = -\frac{\alpha r^2}{2}$$