Homework Solutions 8 (Griffiths Chapter 8)

9 The electric field is $\mathbf{E} = -\frac{1}{\epsilon_0}\sigma\,\hat{\mathbf{z}}$ between the plates, zero elsewhere. $\mathbf{B} = 0$.

(a) Using equation 8.17 for the stress tensor, all off-diagonal elements are zero.

$$\overleftrightarrow{\mathbf{T}} = \frac{\sigma^2}{2\epsilon_0}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & +1 \end{pmatrix}$$

(b) Since $\mathbf{B} = 0$, $\mathbf{S} = 0$. Therefore, using equation 8.20 with the xy-plane as the surface and $d\mathbf{a} = -dxdy\,\hat{\mathbf{z}}$,

$$\mathbf{F} = \oint d\mathbf{a}\cdot\overleftrightarrow{\mathbf{T}} = -\oint dxdy\,T_{zz}\,\hat{\mathbf{z}} = -\frac{\sigma^2 A}{2\epsilon_0}\,\hat{\mathbf{z}}$$

where A is the area of the surface. Therefore the force per unit area is

$$\frac{\mathbf{F}}{A} = -\frac{\sigma^2}{2\epsilon_0}\,\hat{\mathbf{z}}$$

(c) $-T_{zz}$ is also the momentum per unit area per unit time crossing the xy-plane.

(d) The recoil force *is* the momentum transfer per unit time, which is the force per unit area, so it has to be the same as (b).

19 Classical electron:

(a) Use the solutions for a spinning charged sphere, particularly $\mathbf{A}$, from section 5.4.1. Inside, for $r < R$ and $\sigma = e/4\pi R^2$,

$$\mathbf{B} = \boldsymbol{\nabla}\times\left(\frac{\mu_0 R\omega\sigma}{3}r\sin\theta\,\hat{\boldsymbol{\phi}}\right) = \frac{2}{3}\mu_0\sigma R\omega\,\hat{\mathbf{z}} \qquad \mathbf{E} = 0$$

Outside, for $r > R$,

$$\mathbf{B} = \frac{\mu_0\sigma\omega R^4}{3r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}) \qquad \mathbf{E} = \frac{e}{4\pi\epsilon_0\,r^2}\,\hat{\mathbf{r}}$$

The energy in the magnetic field is, for $r < R$,

$$u_B = \frac{1}{2\mu_0} B^2 = \frac{\mu_0 \omega^2 e^2}{72\pi^2 R^2} \quad \Rightarrow \quad U_B = \int_{r<R} dv\, u_B = \frac{\mu_0 e^2 \omega^2 R}{54\pi}$$

For the outside,

$$\begin{aligned} u_B &= \frac{\mu_0 \sigma^2 \omega^2 R^8}{18 r^6} (4\cos^2\theta + \sin^2\theta) \quad \Rightarrow \\ U_B &= \int_{r>R} d\phi d\theta\, \sin\theta\, dr\, r^2\, u_B = \frac{\mu_0 e^2 \omega^2 R}{108\pi} \end{aligned}$$

Now, the electric field, which is nonzero only outside:

$$u_E = \frac{1}{2}\epsilon_0 E^2 = \frac{e^2}{32\pi^2 \epsilon_0 r^4} \quad \Rightarrow \quad U_E = \int_{r>R} dv\, u_E = \frac{e^2}{8\pi\epsilon_0 R}$$

Therefore

$$U = U_E + U_B = \frac{e^2}{8\pi\epsilon_0 R} + \frac{\mu_0 e^2 \omega^2 R}{36\pi}$$

(b) The angular momentum density is nonzero for $r > R$:

$$\boldsymbol{l} = \epsilon_0 [\mathbf{r} \times (\mathbf{E} \times \mathbf{B})] = \frac{\mu_0 e \sigma \omega R^4}{12\pi\epsilon_0 r^5} \sin\theta [r\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}})] = -\frac{\mu_0 e \sigma \omega R^4}{12\pi\epsilon_0 r^4} \sin\theta\, \hat{\boldsymbol{\theta}}$$

Integrating this, we should be careful to write $\hat{\boldsymbol{\theta}}$ in Cartesian coordinates. Only the z-component will give a nonzero result on integration:

$$\mathbf{L} = \frac{\mu_0 e \sigma \omega R^4}{12\pi\epsilon_0} \int_{r>R} dv\, \frac{\sin^2\theta}{r^4} \hat{\mathbf{z}} = \frac{2\mu_0 e \sigma \omega R^3}{9\epsilon_0}\, \hat{\mathbf{z}} = \frac{\mu_0 e^2 \omega R}{18\epsilon_0}\, \hat{\mathbf{z}}$$

(c) If the field angular momentum is equal to the spin,

$$\frac{\mu_0 e^2 \omega R}{18\epsilon_0}\, \hat{\mathbf{z}} = \frac{\hbar}{2} \quad \Rightarrow \quad \omega R = \frac{9\pi\hbar}{\mu_0 e^2}$$

If all the energy is equal to the mass, $U = m_e c^2$, therefore

$$\frac{e^2}{8\pi\epsilon_0 R} \left[1 + \frac{2(\omega R)^2}{9c^2}\right] = mc^2$$

$$R = \frac{e^2\left[1 + \frac{2\left(\frac{9\pi\hbar}{\mu_0 e^2}\right)^2}{9c^2}\right]}{8\pi\epsilon_0 m_e c^2} = 2.95 \times 10^{-11}\,\mathrm{m}$$

$$\omega = \frac{9\pi\hbar}{\mu_0 e^2 R} = 3.13 \times 10^{21}\,\mathrm{rad/s}$$

$$\omega R = 9.23 \times 10^{10}\,\mathrm{m/s} \gg c$$

Clearly not realistic.