Homework Solutions 9 (Griffiths Chapter 9)

17 Use equations (9.101) with $(\tilde{E}_{0_I})_{x,z} = 0$. Item (i) gives $0 = 0$. Using Snell's law, (ii) results in:

$$\frac{1}{v_1}\left(\tilde{E}_{0_I} + \tilde{E}_{0_R}\right)\sin\theta_I = \frac{1}{v_2}\tilde{E}_{0_T}\sin\theta_T \quad \Rightarrow \quad \tilde{E}_{0_I} + \tilde{E}_{0_R} = \tilde{E}_{0_T}$$

That's also what is given by (iii), so that was redundant. Then, (iv) gives:

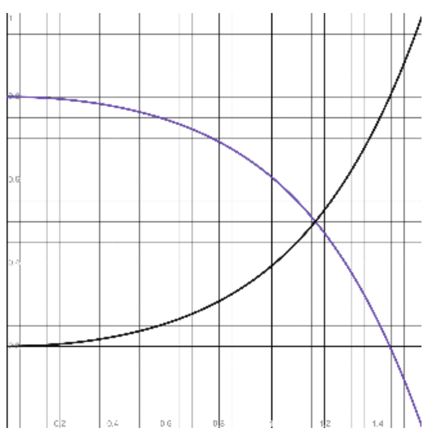
$$\frac{1}{\mu_1 v_1}\left(-\tilde{E}_{0_I} + \tilde{E}_{0_R}\right)\cos\theta_I = -\frac{1}{\mu_2 v_2}\tilde{E}_{0_T}\cos\theta_T \quad \Rightarrow \quad \tilde{E}_{0_I} - \tilde{E}_{0_R} = \alpha\beta\tilde{E}_{0_T}$$

Adding and then subtracting the two equations, we get Fresnel's equations:

$$\tilde{E}_{0_T} = \frac{2}{1+\alpha\beta}\,\tilde{E}_{0_I} \qquad \tilde{E}_{0_R} = \frac{1-\alpha\beta}{1+\alpha\beta}\,\tilde{E}_{0_I}$$

Now, to draw the graph,

$$\alpha\beta = \sqrt{\frac{\beta^2 - \sin^2\theta_I}{\cos^2\theta_I}}$$



The upward curving line is $\tilde{E}_{0_R}/\tilde{E}_{0_I}$; note that since $\alpha\beta > 1$ for $0 < \theta_I < \pi/2$, the real amplitude ratio is the absolute value.

There is no Brewster's angle where $\tilde{E}_{0_R} = 0$. For $\alpha\beta = 1$, we need

$$1 - \left(\frac{v_1}{v_2}\right)^2 \sin^2\theta = \left(\frac{\mu_2 v_2}{\mu_1 v_1}\right)^2 \cos^2\theta$$

For most media, $\mu_1 \approx \mu_2$, so we get $1 \approx (v_2/v_1)^2$, but that just means there is no true interface.

$\alpha = 1$ at normal incidence, so

$$\tilde{E}_{0_T} = \frac{2}{1+\beta}\,\tilde{E}_{0_I} \qquad \tilde{E}_{0_R} = \frac{1-\beta}{1+\beta}\,\tilde{E}_{0_I}$$

which are the same as equation (9.82).

The reflection and transmission coefficients are

$$R = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)^2 \qquad T = \frac{\epsilon_2 v_2}{\epsilon_1 v_1}\left(\frac{2}{1+\alpha\beta}\right)^2 \frac{\cos\theta_T}{\cos\theta_I} = \alpha\beta\left(\frac{2}{1+\alpha\beta}\right)^2$$

$$R + T = \frac{(1-2\alpha\beta+\alpha^2\beta^2)+4\alpha\beta}{(1+\alpha\beta)^2} = \frac{(1+2\alpha\beta+\alpha^2\beta^2)}{(1+\alpha\beta)^2} = 1$$

32 For TM, we want $B_z = 0$ and $E_z = X(x)Y(y)$. The boundary conditions require that $\mathbf{E}^{\|} = 0$, and therefore $E_z = 0$ when $x = 0, x = a, y = 0, y = a$. We have

$$X(x) = A\sin k_x x + B\cos k_x x \qquad Y(y) = C\sin k_y y + D\cos k_y y$$

With the boundary conditions, $B = D = 0$, and we end up with

$$E_x = E_0 \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \qquad m, n = 1, 2, 3, \ldots$$

The cutoff frequencies, and the wave and group velocity results are exactly the same as with TE modes. But because now m and n start from 1, not 0, the lowest TM mode is TM_{11}, while with TE it's TE_{10}. Therefore the frequency ratio is

$$\frac{\omega_{11}}{\omega_{10}} = \frac{c\pi\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}{c\pi\sqrt{\frac{1}{a^2}}} = \sqrt{1+\frac{a^2}{b^2}}$$

37 Spherical waves, involving $j_1(kr - \omega t)$.

(a) Since only $E_\phi \neq 0$, and that doesn't depend on ϕ,

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}E_\phi = 0$$

Then the curl:

$$\boldsymbol{\nabla}\times\mathbf{E} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta E_\phi)\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial}{\partial r}(rE_\phi)\hat{\boldsymbol{\theta}}$$

$$= \frac{2A\cos\theta}{r^2}\left(\cos u - \frac{\sin u}{kr}\right)\hat{\mathbf{r}} - \frac{A\sin\theta}{r}\left(-k\sin u + \frac{\sin u}{kr^2} - \frac{\cos u}{r}\right)\hat{\boldsymbol{\theta}}$$

Next,

$$\boldsymbol{\nabla}\times\mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B} \quad \Rightarrow \quad \mathbf{B} = \int^t dt\,\boldsymbol{\nabla}\times\mathbf{E}$$

$$\mathbf{B} = \frac{2A\cos\theta}{\omega r^2}\left(\sin u + \frac{\cos u}{kr}\right)\hat{\mathbf{r}} + \frac{A\sin\theta}{\omega r}\left(-k\cos u + \frac{\cos u}{kr^2} + \frac{\sin u}{r}\right)\hat{\boldsymbol{\theta}}$$

We now need to check the remaining Maxwell's equations with this $\mathbf{B}$:

$$\boldsymbol{\nabla}\cdot\mathbf{B} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 B_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, B_\theta\right) = 0$$

And then,

$$\boldsymbol{\nabla}\times\mathbf{B} = \frac{1}{r}\left(\frac{\partial}{\partial r}\left(rB_\theta\right) - \frac{\partial}{\partial\theta}B_r\right)\hat{\boldsymbol{\phi}} = \frac{A\sin\theta}{cr}\left(k\sin u + \frac{\cos u}{r}\right)\hat{\boldsymbol{\phi}}$$

which is the same as

$$\mu_0\epsilon_0\frac{\partial}{\partial t}\mathbf{E} = \frac{A\sin\theta}{cr}\left(k\sin u + \frac{\cos u}{r}\right)\hat{\boldsymbol{\phi}}$$

(b) Just grinding through the algebra, the ugly result is

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}$$

$$= \frac{A^2\sin\theta}{\mu_0\omega r^2}\left\{\frac{2\cos\theta}{r}\left[\left(1 - \frac{1}{k^2r^2}\right)\sin u\cos u + \frac{1}{kr}\left(\cos^2 u - \sin^2 u\right)\right]\hat{\boldsymbol{\theta}}\right.$$

$$\left. + \sin\theta\left[\left(-\frac{2}{r} + \frac{1}{k^2r^3}\right)\sin u\cos u + k\cos^2 u + \frac{1}{kr^2}\left(\sin^2 u - \cos^2 u\right)\right]\hat{\mathbf{r}}\right\}$$

Averaging, we have $\langle\cos^2 u\rangle = \langle\sin^2 u\rangle = \frac{1}{2}$ and $\langle\sin u\cos u\rangle = 0$.

$$\mathbf{I} = \langle\mathbf{S}\rangle = \frac{A^2\sin^2\theta}{2\mu_0 c}\frac{1}{r^2}\,\hat{\mathbf{r}}$$

That's the correct direction and r-dependence for a spherical wave.

(c) Total power radiated:

$$\int d\mathbf{a}\cdot\mathbf{I} = \frac{A^2}{2\mu_0 c}\int r^2 dr\sin\theta\, d\theta d\phi\,\frac{\sin^2\theta}{r^2} = \frac{4\pi A^2}{3\mu_0 c}$$

41 Total internal reflection:

(a) We have $\tilde{\mathbf{E}}_T = \tilde{\mathbf{E}}_{0_T} e^{i(\mathbf{k}_T\cdot\mathbf{r}-\omega t)}$. Then,

$$\mathbf{k}_T\cdot\mathbf{r} = k_T(x\sin\theta_T + z\cos\theta_T) = kx + i\kappa z \quad\Rightarrow\quad \tilde{\mathbf{E}}_T = \tilde{\mathbf{E}}_{0_T} e^{-\kappa z} e^{i(kx-\omega t)}$$

(b) With complex α, β,

$$R = \left|\frac{\tilde{E}_{0_R}}{\tilde{E}_{0_I}}\right|^2 = \left|\frac{\alpha-\beta}{\alpha+\beta}\right|^2$$

In our case β is real, but α is imaginary: $\alpha = i|\alpha|$. Therefore, if $\beta + i|\alpha| = \mathcal{Z}$, then

$$R = \left|\frac{-\mathcal{Z}^*}{\mathcal{Z}}\right|^2 = 1$$

since that is true for any complex number.

(c) The other polarization, and again with β real and α imaginary:

$$R = \left|\frac{1-\alpha\beta}{1+\alpha\beta}\right|^2 = 1$$

(d) Using the solutions to 9.17 and (a) here, the real part of $\tilde{\mathbf{E}}_T$ is

$$\mathbf{E} = E_0 e^{-\kappa z}\cos(kx-\omega t+\delta)\,\hat{\mathbf{y}}$$

It seems Griffiths wants us to choose the phase $\delta = 0$. We'll do so. The complex magnetic field is

$$\tilde{\mathbf{B}} = \frac{1}{v_2}\tilde{E}_{0_T} e^{-\kappa z} e^{i(kx-\omega t)}\left(-i\frac{c\kappa}{\omega n_2}\hat{\mathbf{x}} + \frac{ck}{\omega n_2}\hat{\mathbf{z}}\right)$$

Its real part is

$$\begin{aligned}\mathbf{B} &= \frac{cE_0}{v_2\omega n_2}\mathrm{Re}\left\{[\cos(kx-\omega t) + i\sin(kx-\omega t)]\,[-i\kappa\,\hat{\mathbf{x}} + k\,\hat{\mathbf{z}}]\right\}\\ &= \frac{E_0}{\omega}e^{-\kappa z}\left[\kappa\sin(kx-\omega t)\,\hat{\mathbf{x}} + k\cos(kx-\omega t)\,\hat{\mathbf{z}}\right]\end{aligned}$$

(e) Grind through the equations:

$$\nabla \cdot \mathbf{E} = \frac{\partial}{\partial y} E_y = 0$$

$$\nabla \times \mathbf{E} = \kappa E_0 e^{-\kappa z} \cos(kx - \omega t)\, \hat{\mathbf{x}} - k E_0 e^{-\kappa z} \sin(kx - \omega t)\, \hat{\mathbf{z}} = -\frac{\partial}{\partial t} \mathbf{B}$$

$$\nabla \cdot \mathbf{B} = \frac{\partial}{\partial x} B_x + \frac{\partial}{\partial z} B_z = 0$$

$$\nabla \times \mathbf{B} = (k^2 - \kappa^2) \frac{E_0}{\omega} e^{-\kappa z} \sin(kx - \omega t)\, \hat{\mathbf{y}}$$

Here, note that $k^2 - \kappa^2 = (n_2 \omega / c)^2 = \omega^2 \mu_2 \epsilon_2$. So

$$\nabla \times \mathbf{B} = \mu_2 \epsilon_2 E_0 \omega\, e^{-\kappa z} \sin(kx - \omega t)\, \hat{\mathbf{y}} = \mu_2 \epsilon_2 \frac{\partial}{\partial t} \mathbf{E}$$

(f) Poynting vector:

$$\begin{aligned} \mathbf{S} &= \frac{1}{\mu_2} \mathbf{E} \times \mathbf{B} \\ &= \frac{E_0^2}{\mu_2 \omega} e^{-2\kappa z} [k \cos^2(kx - \omega t)\, \hat{\mathbf{x}} - \kappa \sin(kx - \omega t) \cos(kx - \omega t)\, \hat{\mathbf{z}}] \end{aligned}$$

Averaging, $\langle \cos^2(kx - \omega t) \rangle = \frac{1}{2}$ and $\langle \sin(kx - \omega t) \cos(kx - \omega t) \rangle = 0$, so

$$\mathbf{I} = \langle \mathbf{S} \rangle = \frac{E_0^2 k}{2 \mu_2 \omega}\, e^{-2\kappa z}\, \hat{\mathbf{x}}$$

Nothing is transmitted in the z-direction.