Phys 191 Activity Solutions 1: Complex SHO

Recall the simple harmonic oscillator: a mass m on a spring with stiffness k. (No gravity, moving in the x direction.)

1. Write $\sum \vec{F} = m\vec{a}$ for this system. The only force is the spring force, $F_{sp,x} = -kx$; express it as a *differential equation* in terms of x and its time derivatives.

Answer: Acceleration is the second time derivative of position, so

$$-kx = m\frac{d^2}{dt^2}x$$

2. Often the easiest way to solve a differential equation is to guess and check. Guess that $x(t) = Ae^{\kappa t}$, where A and κ are constants. Exponential growth or decay may seem a strange guess for an oscillator—exponentials don't oscillate. Try it anyway. Plug in and see what you can find out about A and/or κ.

Answer: Taking the second derivative,

$$-k\left(Ae^{\kappa t}\right) = m\left(A\kappa^2 e^{\kappa t}\right) \quad \Rightarrow \quad -k = \kappa^2$$

Therefore $\kappa = \pm\sqrt{-k} = \pm ik$. A can be anything.

You should find that κ must be an imaginary number. But since imaginary numbers exist, let's use them to our advantage. We just need to know what $e^{i\omega t}$ *means* when ωt is real. And we already know the solution to the harmonic oscillator—it's sines and cosines—so let's guess (again) that

$$e^{i\omega t} = A\cos\omega t + B\sin\omega t$$

3. If this is true at all times, it is true for $t = 0$. What does that tell you about A and B?

If this is true at all times, it's also true for the derivative of the equation. Now can you solve for both A and B?

Answer: At $t = 0$, $\sin\omega t = 0$ and $\cos\omega t = 1$, and $e^{i\omega t} = 1$. Therefore, $A = 1$.

With the derivative,

$$i\omega e^{i\omega t} = -A\omega\sin\omega t + B\omega\cos\omega t$$

at $t = 0$, this gives $i\omega = B\omega$. Therefore, $B = i$

Besides real and imaginary numbers, there are *complex* numbers $z = x + iy$, where x and y are both real numbers.

4. What is e^z in terms of x and y? Simplify until you find the real and imaginary parts of e^z. [Note: z is a traditional name for a complex number, but here x, y, and z are just variable names, not position coordinates.]

Answer: Plug it in:

$$e^{x+iy} = e^x e^{iy} = e^x \left(\cos y + i \sin y\right)$$