1. (80 points) Consider the simple harmonic oscillator with an added drag force,

$$F_{d,x} = -bv_x = -b\frac{dx}{dt}$$

(This is the simplest type of drag, called "viscous drag." This system is called the "damped oscillator.")

Try an exponential solution for the resulting differential equation.

There are two important cases and one edge case: κ has two complex solutions with non-zero imaginary parts; κ has two real solutions (zero imaginary parts); and κ has only a single real solution. The edge case divides the two important cases. Write the conditions for satisfying each scenario.

When κ has two real solutions, what does $x(t)$ look like? Write an equation and sketch a graph.

When κ has two complex solutions, what does $x(t)$ look like? Write an equation and sketch a graph.

Answer: Writing down $\sum F_x = m\frac{d^2}{dt^2}x$, we end up with

$$m\frac{d^2}{dt^2}x + b\frac{dx}{dt} + kx = 0$$

Trying $x = Ae^{\kappa t}$, we get

$$m\kappa^2 Ae^{\kappa t} + b\kappa Ae^{\kappa t} + kAe^{\kappa t} = 0 \quad \Rightarrow \quad m\kappa^2 + b\kappa + k = 0$$

The quadratic equation has solutions

$$\kappa_{1,2} = \frac{1}{2m}\left(-b \pm \sqrt{b^2 - 4mk}\right)$$

Therefore, we have two real solutions when $b^2 > 4mk$, so

$$x(t) = Ae^{\kappa_1 t} + Be^{\kappa_2 t}$$

Both $\kappa_1 < 0$ and $\kappa_2 < 0$, so the solutions are pure exponential decays with no oscillation.

There is one real solution $\kappa = -b/2m < 0$ when $b^2 = 4mk$. In a more advanced math course, you will learn that there is a second solution that goes like $te^{\kappa t}$. You don't have to include that in your answers, but if were to do that, you would have

$$x(t) = Ae^{\kappa t} + Bte^{\kappa t}$$

This is also a decaying, non-oscillating solution.

When $b^2 < 4mk$, we get solutions with a nonzero imaginary part:

$$\kappa_{1,2} = -\alpha \pm i\omega \qquad \text{with} \qquad \omega = \sqrt{4mk - b^2}/2m \quad \text{and} \quad \alpha = b/2m > 0$$

$$x(t) = Ae^{-\alpha t}e^{i\omega t} + Be^{-\alpha t}e^{-i\omega t} = Ce^{-\alpha t}\cos(\omega t + \delta)$$

These are sinusoidal oscillations multiplied by an exponential decay envelope.

2. (20 points) Consider the ordinary (undamped) simple harmonic oscillator, with motion

$$x(t) = A\cos(\omega t + \delta)$$

Explain the physical meanings of A, ω, and δ, and how they relate to the motion of the oscillator.

Derive (don't just state) equations for the frequency f and period T of the oscillator. *Hint:* Use the periodicity of the cosine function.

Answer: A is the *amplitude*; x oscillates between a maximum of $+A$ and a minimum of $-A$.

ω is the angular frequency of the motion. Since the cosine function has a period of 2π, this means that $2\pi/T = \omega$ and $2\pi f = \omega$.

δ is just a phase; it's determined by initial conditions. After all, it's not always the case that $x = +A$ at $t = 0$.