PHYS 191 Activity Solutions 2: The Wave Equation

The Wave Equation (for transverse waves on a string)

Imagine a string under tension T as a sequence of masses m, equally spaced by ℓ, connected by springs.

1. Assume the string is along the x axis. When a transverse wave passes along the string, the masses will be slightly displaced in the y direction.

Draw a free-body diagram for one particular mass. (Note that when a wave passes, the string is *not* in static equilibrium!)

Answer: The forces acting on a mass i will be from the spring that connects it to masses $i+1$ and $i-1$. Both forces have x and y components.

2. Label the y displacements of each mass y_i, where i is the index of the mass along the string. Find the angle of each tension force in terms of y_i, $y_{i\pm1}$, and ℓ.

Answer: The angle between the spring and the x-axis on either side ($+$ and $-$ sides) is defined by $\tan\theta_\pm = (y_{i\pm1} - y_i)/\ell = \pm\Delta y_\pm/\ell$. The tension forces are directed along the springs.

3. Assuming that the string is only displaced slightly—the angles are small, the x-acceleration is negligible ($a_x = 0$), and the tension T is constant—calculate the y-component of net force and set up $\sum \vec{F} = m\vec{a}$ for the selected mass. Simplify as much as possible: remember that for small angles $\sin x \approx \tan x \approx x$.

Answer: The transverse tensions are

$$T_{y\pm} = T\sin\theta_\pm \approx T\tan\theta_\pm = T\left(\frac{y_{i\pm1} - y_i}{\ell}\right) = T\left(\frac{\pm\Delta y_\pm}{\ell}\right)$$

The total forces along the y-axis are then

$$T_{y+} + T_{y-} = T\left(\frac{\Delta y_+ - \Delta y_-}{\ell}\right) = m\frac{d^2x}{dt^2}$$

4. We can take another limit, as the string becomes continuous: m and ℓ become small with $m/\ell = \mu = \text{const}$. In this limit the ratios of differences become derivatives, and y_i becomes a continuous function $y(x,t)$. Convert $\sum \vec{F} = m\vec{a}$ into a differential equation for $y(x,t)$.

Answer: We end up with

$$T\left(\frac{\Delta y_+ - \Delta y_-}{\ell^2}\right) = \mu\frac{d^2x}{dt^2}$$

Now, $\ell = \Delta x$. Therefore, the left side of the equation becomes

$$T\left[\frac{\Delta(\Delta y)}{\Delta x^2}\right] \to T\frac{\partial^2 y}{\partial x^2}$$

Properly speaking, the time derivatives re also partial derivatives here, so the equation is a standard 1D wave equation:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

with the wave speed $v = \sqrt{T/\mu}$.

Traveling Waves

A *traveling wave* is a wave $y(x,t)$ that keeps a constant shape while it moves at constant speed v toward the right. This can be represented by

$$y(x,t) = f(x - vt)$$

where $f(x)$ is the shape of the wave at $t = 0$.

5. Sketch a random traveling wave at $t = 0$, and at a slightly later time. What direction is this wave moving? Argue that the speed of the wave is v.

Answer: If $v > 0$, this wave will be traveling toward the right. Any feature of the wave will be reproduced exactly a distance d to the right on the $x - axis$ after a time $t = d/v$ passes. That's exactly what we mean by speed.

6. Show that a traveling wave solves the wave equation if it has a specific speed. (You don't need to know the derivative of f, just call it f', and remember the chain rule.) What is the speed of a traveling wave on a string?

Answer: Put the solution into the wave equation we found:

$$\frac{\partial^2 f(x-vt)}{\partial x^2} = f''(x-vt)$$

$$\frac{1}{v^2}\frac{\partial^2 f(x-vt)}{\partial x^2} = \frac{1}{v^2}v^2 f''(x-vt) = f''(x-vt)$$

Therefore the wave equation will be satisfied.

We have already found the speed of a wave on a string: $v = \sqrt{T/\mu}$.