

PHYS 191 Exam 1

In 2D, waves radiating away from a point source cannot be described by sines and cosines. The equation for waves that depend only on r , the radial distance to the source, and not on ϕ , the polar angle, is

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial f(r, t)}{\partial r} \right) = \frac{1}{v^2} \frac{\partial^2 f(r, t)}{\partial t^2}$$

where $f(r, t)$ is the wave, and v is the speed of the wave.

1. (10 points) Show that if you try a solution like $f = A \cos(kr - \omega t + \delta)$ (alternatively, $f = Ae^{i(kr - \omega t + \delta)}$), it does *not* solve the 2D radial wave equation for any interesting value of k . (Unlike 1D standing waves in your assignment, where solutions existed for a restricted set of k values.)

Answer: The left hand side:

$$\frac{A}{r} \frac{\partial}{\partial r} [-kr \sin(kr - \omega t + \delta)] = -\frac{A}{r} [k \sin(kr - \omega t + \delta) + k^2 r \cos(kr - \omega t + \delta)]$$

This can't be made equal to the right hand side

$$\frac{A}{v^2} \frac{\partial}{\partial t} [\omega \sin(kr - \omega t + \delta)] = -\frac{A}{v^2} \omega^2 \cos(kr - \omega t + \delta)$$

The only way to make these equal is for $k = 0$ and $\omega = 0$ for $f = A$, which is not an interesting solution.

2. (40 points) An approximate form for a 2D wave, valid for $r \gg \lambda$, is

$$f(r, t) \approx A r^p \cos(kr - \omega t + \delta)$$

with p a constant. Show that $f(r, t)$ is a good approximate solution to the 2D wave equation for $r \gg \lambda$. *Hint:* When $r \gg \lambda$, which is true? $k^2 r^p \gg kr^{p-1}$ or $k^2 r^p \ll kr^{p-1}$?

Answer: Again, plug it in and do the derivatives:

$$\begin{aligned} \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial f}{\partial r} \right) &= \frac{A}{r} \frac{\partial}{\partial r} [r(p r^{p-1} \cos(kr - \omega t + \delta) - k r^p \sin(kr - \omega t + \delta))] \\ &= A [p^2 r^{p-2} \cos(kr - \omega t + \delta) - p k r^{p-1} \sin(kr - \omega t + \delta) \\ &\quad - (p+1) k r^{p-1} \sin(kr - \omega t + \delta) - k^2 r^p \cos(kr - \omega t + \delta)] \\ \frac{1}{v^2} \frac{\partial^2 f(r, t)}{\partial t^2} &= \frac{A}{v^2} \frac{\partial}{\partial t} [\omega r^p \sin(kr - \omega t + \delta)] = -\frac{A \omega^2}{v^2} r^p \cos(kr - \omega t + \delta) \end{aligned}$$

These don't seem that similar, except that the last term on the spatial derivative part looks like it could be the time derivative part. But notice that we want the equality to

hold as $r \gg \lambda$. Since $k = 2\pi/\lambda$, this is equivalent to saying that $kr \gg 1$. Therefore,

$$kr^{p-1} = \frac{k^2 r^p}{kr} \ll k^2 r^p \quad \text{and} \quad r^{p-2} = \frac{k^2 r^p}{(kr)^2} \ll k^2 r^p$$

That means that the first three terms in the spatial derivative result are negligible compared to the fourth:

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial f}{\partial r} \right) \approx -Ak^2 r^p \cos(kr - \omega t + \delta)$$

Then,

$$-\frac{A\omega^2}{v^2} r^p \cos(kr - \omega t + \delta) = -Ak^2 r^p \cos(kr - \omega t + \delta)$$

provided that $v = \omega/k$, which is just the usual relationship with basic waves.

3. (10 points) The best approximate solution will be that where the first neglected term in the left hand side of the 2D wave equation (the spatial derivative part) is of the order of $(\lambda/r)^2$ rather than just λ/r —something small squared rather than just something small. What is the value of p that makes this happen? *Hint:* you want to make sure that the term with $\sin(kr - \omega t + \delta)$ in your left hand side is actually zero.

Answer: The first term neglected that was of the order λ/r is the term with the sines:

$$\begin{aligned} -Apkr^{p-1} \sin(kr - \omega t + \delta) - A(p+1)kr^{p-1} \sin(kr - \omega t + \delta) = \\ -A(2p+1)kr^{p-1} \sin(kr - \omega t + \delta) = 0 \end{aligned}$$

To make this zero, $2p+1=0$, therefore $p = -\frac{1}{2}$.

4. (20 points) In a simple harmonic oscillator, the total energy is proportional to the amplitude squared: $E_T \propto |A|^2$. A wave is a traveling oscillation, so the energy transmitted by wave through a location is proportional to the amplitude of the wave at that location. Wavefronts in 2D from a point source spread out in concentric circles, distributing their energy over the circumference of those circles. That is, the energy transmitted through any point a distance r from the source is proportional to the amplitude of the wave at r , therefore, the total energy transmitted over a circle with radius r is

$$E_{\text{circle}} \propto \int_{\text{circle}} d(\text{arc length}) |\text{amplitude}|^2$$

Remember that the infinitesimal arc length on a circle is $r d\phi$ in polar coordinates.

If energy is conserved for 2D waves (and it should be!), then the total energy transmitted through a circle centered on the source should not depend on r . In that case, what value must p have to obey energy conservation? (This is the same p as in the previous questions.)

Answer: $f(r, t)$ does not depend on the angle ϕ . At any given r , f will oscillate up and down with amplitude Ar^p . Therefore, the energy transmitted through a circle will be

$$E_{\text{circle}} \propto \int_0^{2\pi} (r d\phi) A^2 r^{2p} = 2\pi A^2 r^{2p+1}$$

If energy is conserved, this result cannot depend on r . Therefore $2p + 1 = 0$ and $p = -\frac{1}{2}$ again. This should not be a surprise.

5. (20 points) For spherically symmetric waves emanating from a point source in 3D,

$$f(r, t) \approx A r^q \cos(kr - \omega t + \delta)$$

where r is the distance to the source, and the approximation is valid when $r \gg \lambda$. The energy will now be transmitted through infinitesimal surface areas, not infinitesimal arcs. In that case, what do you expect q will be? Explain.

Answer: In 3D, the energy from a point source will be spread over a sphere's area rather than over a circle's circumference:

$$E_{\text{sphere}} \propto \int_{\text{sphere}} d(\text{area}) |\text{amplitude}|^2 = 4\pi A^2 r^{2p+2}$$

If that is not to depend on r , we must have $p = -1$.

Note on Bessel functions

In more advanced coursework, you will encounter the counterparts of sines and cosines for the spatial part of waves, in 2D polar coordinates, that are known as Bessel functions. These are oscillating functions where the amplitude decreases as r increases, and when $r \gg \lambda$, these go like

$$J_0(kr) \approx (\text{const}) \frac{1}{\sqrt{kr}} \cos(kr + \delta)$$

In 3D, we have spherical Bessel functions, which when $r \gg \lambda$, behave like

$$j_0(kr) \approx (\text{const}) \frac{1}{kr} \cos(kr + \delta)$$

When you multiply these with the $\cos(\omega t)$ time dependence for waves, you get the approximate forms for $f(r, t)$ in the exam.