PHYS 191 Activity 3: Resonance

As we saw on the previous problem set, standing waves can exist only at certain frequencies. These are called *resonant frequencies* and they occur in many systems; in fact, the SHO is the simplest example of a resonant system.

1. Write $\sum \vec{F} = m\vec{a}$ for an undamped harmonic oscillator with an external, time-dependent driving force, $F_{ext}(t)$.

Answer: This is straightforward:

$$F_{ext,x} - kx = m\frac{d^2x}{dt^2}$$

2. We can try an exponential solution for this equation again. In order to do so, we will need to assume that the force is also sinusoidal, e.g. $F_{ext}(t) = F_0 \cos(\omega t)$. Then we can assume that the system is the real part of a complex equation with $F_{ext}(t) = F_0 e^{i\omega t}$.

Now solve for $x(t) = Ae^{i\omega t}$. Unlike before, we can solve for A!

Answer: Plug everything in:

$$m\frac{d^2 Ae^{i\omega t}}{dt^2} + kAe^{i\omega t} = F_0 e^{i\omega t} \quad \Rightarrow \quad -Am\omega^2 + kA = F_0 \quad \Rightarrow \quad A = \frac{F_0}{k - m\omega^2}$$

3. Sketch a graph of A vs ω. What happens at very low frequencies? Very high? Is there a special in-between frequency—what is it?

Answer: At low frequencies ($\omega \to 0$), we have $A \to F_0/k$. That corresponds to the spring just compressing enough to cancel a time-independent (since $\omega = 0$) external force. Then, A gradually increases as ω increases.

As $\omega \to \sqrt{k/m}$, we get a response that goes out of control: $A \to \infty$. That's resonance. When you drive a system at its natural frequencies—remember that $\omega = \sqrt{k/m}$ for a SHO—that's when it responds most strongly. In fact, here, the response blows up, since this is an undamped SHO with no way of getting rid of energy pumped into the system.

As ω grows beyond the resonance blow-up, A becomes negative (nothing wrong with that), and eventually, as $\omega \to \infty$, the driving frequency becomes completely mismatched with the natural frequency. Everything happens too fast, and the spring has no time to respond at all: $A \to 0$.