Solutions 3; Phys 191 Name ______________

**1. (40 points)** Write $\sum \vec{F} = m\vec{a}$ for a *damped* harmonic oscillator (As in Assignment 1) with an external, time-dependent driving force, $F_{ext}(t) = F_0 e^{i\omega t}$. Solve for $x(t) = Ae^{i\omega t}$ and find $A$.

**Answer:** Adding the forces leads to a differential equation:

$$m\frac{d^2}{dt^2}x + b\frac{dx}{dt} + kx = F_0 e^{i\omega t}$$

Plugging in the exponential solutions for $x(t)$, we get

$$-m\omega^2 Ae^{i\omega t} + ib\omega Ae^{\omega t} + kAe^{i\omega t} = F_0 e^{i\omega t} \quad \Rightarrow \quad A\left(-m\omega^2 + ib\omega + k\right) = F_0$$

Therefore

$$A = \frac{F_0}{-m\omega^2 + ib\omega + k}$$

**2. (30 points)** Sketch a graph of $|A|$ vs $\omega$. Is there resonance here as in the undamped oscillator? What strikes you as different in the damped case?

**Answer:** For a complex number $z$, $|z|^2 = zz^*$, where $z^*$ is its complex conjugate.

$$|A| = \sqrt{\frac{F_0}{-m\omega^2 + ib\omega + k}\frac{F_0}{-m\omega^2 - ib\omega + k}} = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + b^2\omega^2}}$$

If you plot this, you will notice that $|A|$ goes through a maximum, just as in the undamped case. The energy loss associated with the damping when $b > 0$, however, prevents $|A| \to \infty$; the resonance is a mere maximum.

**3. (30 points)** The maxima and minima of $|A|(\omega)$ can be obtained by finding the derivative of $|A|$ with respect to $\omega$ and the values of $\omega$ where this derivative is zero. Find these maxima and minima. Check whether you get the undamped SHO results when $b = 0$.

**Answer:** Do the derivative:

$$\frac{d}{d\omega}\left(\frac{F_0}{\sqrt{(k - m\omega^2)^2 + b^2\omega^2}}\right) = -\frac{F_0}{2}\left[(k - m\omega^2)^2 + b^2\omega^2\right]^{-3/2}\left[2(k - m\omega^2)(-2m\omega) + 2b^2\omega\right]$$

For this to be zero,

$$\left[2(k - m\omega^2)(-2m\omega) + 2b^2\omega\right] = 0 \quad \Rightarrow \quad \omega = 0 \quad \text{or} \quad \omega = \sqrt{\frac{1}{m}\left(k - \frac{b^2}{2m}\right)}$$

When $b = 0$, we get $\omega = \sqrt{k/m}$, which is the resonant frequency for an undamped SHO.