Solutions 4; Phys 191 Name ______________

1. (50 points) Calculate the electric field of a solid flat disk of radius R with uniform *surface* charge density σ, as measured at a height z above the center of the disk. (For a bit of surface area dA, the bit of charge is $dq = \sigma dA$.) *Hint:* Divide up the disk into infinitesimally thin rings and integrating up the electric field of the rings, using the ring result we obtained in class.

Take the limit $R \to \infty$ and verify that you get the result for an infinite plane, which is $\vec{E} = \pm\sigma/2\epsilon_0\,\hat{\mathbf{z}}$. ($\pm$ depending on whether you're above or below the plane.)

Answer: Each ring of radius r will have area $dA = 2\pi r dr$ and charge $dq = \sigma(2\pi r dr)$. For $z > 0$, it produces an electric field

$$d\vec{E} = \frac{k\,dq\,z}{(z^2+r^2)^{3/2}}\hat{\mathbf{z}} = \frac{k\sigma(2\pi r dr)z}{(z^2+r^2)^{3/2}}\hat{\mathbf{z}}$$

Integrating over the disk,

$$\vec{E} = \int_{\text{disk}} d\vec{E} = 2\pi k\sigma z\hat{\mathbf{z}}\int_0^R dr\,\frac{r}{(z^2+r^2)^{3/2}} = -\left.\frac{2\pi k\sigma z\hat{\mathbf{z}}}{(z^2+r^2)^{1/2}}\right|_0^R = 2\pi k\sigma z\left(\frac{1}{z} - \frac{1}{\sqrt{z^2+R^2}}\right)\hat{\mathbf{z}}$$

When $R \to \infty$ and $z > 0$,

$$\vec{E} \to 2\pi k\sigma z\left(\frac{1}{z}\right)\hat{\mathbf{z}} = 2\pi\frac{1}{4\pi\epsilon_0}\sigma\hat{\mathbf{z}} = \frac{\sigma}{2\epsilon_0}\hat{\mathbf{z}}$$

which is as it should be.

2. (50 points) Calculate the electric field of a solid flat square of side length b with uniform *surface* charge density σ, as measured at a height z above the center of the square. (For a bit of surface area dA, the bit of charge is $dq = \sigma dA$.) *Hint:* Divide up the disk into thin strips (lines) along the x-axis and integrating up the electric field of the lines, using the line result we obtained in class.

Take the limit $b \to \infty$ and verify that you get the result for an infinite plane, which is $\vec{E} = \pm\sigma/2\epsilon_0\,\hat{\mathbf{z}}$. ($\pm$ depending on whether you're above or below the plane.)

Answer: Each strip of length b, centered at $y = 0$ and $-b/2 \le x \le b/2$, will have area $dA = b\,dx$ and charge $dq = \sigma(b\,dx)$. For $z > 0$, it produces an electric field

$$d\vec{E} = \left(\frac{k\sigma b\,dx}{r\left[r^2 + \left(\frac{b}{2}\right)^2\right]^{1/2}}\right)\hat{\mathbf{r}}$$

with $r = \sqrt{x^2+z^2}$ and $\hat{\mathbf{r}} = z/r\,\hat{\mathbf{z}} - x/r\,\hat{\mathbf{x}}$.

Due to symmetry, the total $E_x = 0$, so the result will be

$$\begin{aligned}
\vec{E} &= \int_{\text{square}} d\vec{E} = k\sigma b z\,\hat{\mathbf{z}} \int_{-b/2}^{b/2} \frac{dx}{(x^2+z^2)\left[x^2+z^2+\left(\frac{b}{2}\right)^2\right]^{1/2}} \\
&= 4k\sigma \tan^{-1}\left(\frac{b^2}{4z(z^2+b^2/2)^{1/2}}\right)\hat{\mathbf{z}}
\end{aligned}$$

This is ugly. But when $b \to \infty$, it gives the proper limit:

$$\vec{E} \to 4k\sigma(\tan^{-1}\infty)\hat{\mathbf{z}} = \frac{4\sigma}{4\pi\epsilon_0}\frac{\pi}{2}\hat{\mathbf{z}} = \frac{\sigma}{2\epsilon_0}\hat{\mathbf{z}}$$