Phys 191 Activity 6: Electric Potential and Field

Recall that the work done *by* a force is

$$W = \int \vec{F} \cdot d\vec{r}$$

If (and only if) the value of that integral does not depend on the path taken, then

$$W = -\Delta U = -(U(\vec{r}_f) - U(\vec{r}_i))$$

defines the potential energy function $U(\vec{r})$.

1. Does the electric work done on a point charge q by a stationary point charge Q depend on the path that q follows? Explain.

If the answer is no, calculate ΔU when q is moved from one point to another. Do this by picking a convenient path.

Answer: Path independence is equivalent to saying that the work integral over a *closed loop* is zero:

$$\oint \vec{F} \cdot d\vec{r} = 0$$

For the point charge, choose a loop where we move q radially away from Q, then move sideways while keeping the radial distance between the charges constant, then radially come back to the original distance, and finally go back to the original location while keeping the radial distance constant again. In the constant radius movements, $\vec{F} \perp d\vec{r} = 0$, so only the lines where the radius changes will contribute the work. In polar coordinates centered on Q, this is then

$$kqQ \left[\int_{r_i}^{r_f} dr\, \frac{1}{r^2} + \int_{r_f}^{r_i} dr\, \frac{1}{r^2} \right] = 0$$

Every loop (in 3D as well as 2D) can be made up by combining such loops, so the electrostatic work over closed paths is always zero. Therefore a potential energy function U exists.

To get ΔU, take the combination of constant r movement (no work) and then a radial movement to get to the final location

$$\Delta U = -W = -\int_{\vec{r}_i}^{\vec{r}_f} \vec{F} \cdot d\vec{r} = 0 - kqQ \int_{r_i}^{r_f} dr\, \frac{1}{r^2} = kqQ \left(\frac{1}{r_i} - \frac{1}{r_f} \right)$$

This suggests that $U = kqQ/r$.

2. Will the above answers change if there are many stationary charges Q_i, or even a stationary charge distribution (like a line or plane of charge)?

Answer: No change. Since the electric field is a *sum* of the electric fields produced by individual charges, all of the above will generalize to any static configuration of charges.

3. If we pick a reference point $\mathcal{O}$ where $U(\mathcal{O}) = 0$, then the potential energy anywhere else can be written as

$$U(\vec{r}) = -\int_{\mathcal{O}}^{\vec{r}} \vec{F} \cdot d\vec{r}'$$

Since U is an integral of $\vec{F}$, we should be able to find $\vec{F}$ using derivatives of U. How? (Hint: think about finding dU for a small displacement along either the x, y, or z axis.)

Answer: Let's move a tiny amount dx parallel to the x-axis. Then, $\vec{r}_f = \vec{r}_i + dx\,\hat{\mathbf{x}}$.

$$dU = -\int_{\mathcal{O}}^{\vec{r}_f} \vec{F} \cdot d\vec{r}' + \int_{\mathcal{O}}^{\vec{r}_f} \vec{F} \cdot d\vec{r}' = -\int_{\vec{r}_i}^{\vec{r}_f} \vec{F} \cdot d\vec{r}' = -\int_{\vec{r}_i}^{\vec{r}_f} F_x dx = -F_x\, dx$$

Remember, we kept $dy = dz = 0$. Therefore dU/dx is actually $\partial U/\partial x$. The argument doesn't depend on the axis label, therefore

$$F_x = -\frac{\partial U}{\partial x}, \quad F_y = -\frac{\partial U}{\partial y}, \quad F_z = -\frac{\partial U}{\partial z}, \qquad \text{or} \quad \vec{F} = -\vec{\nabla} U$$

The force is $-$ the gradient of the potential energy.

4. How should we define the electric potential V? (It should be related to the electric potential *energy* U in the same way that the electric *field* relates to the *force*.)

How are V and $\vec{E}$ related? How can we convert between them in each direction?

Answer: Since the electrostatic force $\vec{F} = q\vec{E}$, the same relationship will apply to the voltage or electric potential:

$$U = qV \qquad \text{and} \qquad \vec{E} = -\vec{\nabla} V$$