# PHYS 191 Activity 7: RC Circuits

Draw a single-loop circuit including a switch, voltage source $V_s$, resistance $R$, and capacitor $C$. At time $t = 0$, the switch will be closed.

**1.** Write down the loop equation relating the voltages across the source, resistance, and battery after the switch is closed. Then write the loop equation in the form of a differential equation for $Q$, the charge on the capacitor. Remember that $Q = CV_C$ and $V_R = RI$. What is the relationship between $I$ and $Q$?

**Answer:** The loop equation is $V_s = V_C + V_R$. Since the current in the circuit is entirely due to the charging and discharging of the capacitor, $I = \frac{d}{dt}Q$. Putting these together,

$$V_s = \frac{1}{C}Q + R\frac{dQ}{dt}$$

**2.** Say $V_s = 0$: this will be a discharge circuit for the capacitor. Solve the differential equation for the circuit in this case by direct integration—rearrange the equation until you have $\int dt\,(\text{something}) = \int dQ\,(\text{something})$ and you can take the integrals. Then, use the initial condition that at $t = 0$, the initial charge on the capacitor was $Q_0$.

**Answer:** Rearranging the equation,

$$-\frac{1}{C}Q = R\frac{dQ}{dt} \quad \Rightarrow \quad -\frac{1}{C}dt = R\frac{dQ}{Q} \quad \Rightarrow \quad -\frac{1}{C}\int_{t_i}^{t} dt = R\int_{Q_i}^{Q} \frac{dQ}{Q}$$

Doing the integrals,

$$\frac{1}{C}(t - t_i) = R(\ln Q - \ln Q_i) \quad \Rightarrow \quad \frac{Q}{Q_i} = \exp\left(-\frac{t - t_i}{RC}\right) \quad \Rightarrow \quad Q = Q_0\, e^{-t/RC}$$

The charge exponentially decays.

**3.** Say $V_s = V_0$, a constant. The solution of a *linear* differential equation can be written as $Q = Q_p + Q_h$. Here, $Q_h$ is the solution to the equation with $V_s = 0$, including the undetermined multiplying constant. We already have this. We just need $Q_p$, a "particular solution" with no undetermined constants. What would be $Q_p$ for the circuit with a constant voltage source? Try to guess—in this case it's easy, even trivial.

**Answer:** A constant will work: $Q_p = CV_0$.

**4.** Write down the full solution for the case $V_s = V_0$, with no undetermined constants, using the initial condition that at $t = 0$, $Q = 0$.

**Answer:** The full solution is

$$Q = Q_p + Q_h = CV_0 + Q_0\, e^{-t/RC}$$

At $t = 0$, we have $0 = CV_0 + Q_0$, therefore $Q_0 = -CV_0$. The solution is

$$Q = CV_0 \left(1 - e^{-t/RC}\right)$$

This describes a circuit gradually charging a capacitor up to its maximum charge of $CV_0$.

**5.** We now have an AC voltage, with $V_s = V_0\, e^{i\omega t}$. What would $Q_p$ be in this case? Remember the early weeks of the semester.

**Answer:** Try an exponential solution $Q_p = A\, e^{i\omega t}$.

$$V_0\, e^{i\omega t} = \frac{1}{C}\, A\, e^{i\omega t} + iR\omega\, A\, e^{i\omega t} \qquad \Rightarrow \qquad A = \frac{CV_0}{1 + iRC\omega}$$

Note that $A$ is complex and $\omega$-dependent, indicating a frequency-dependent amplitude and also a frequency-dependent phase difference between the voltage and the charge (and current).

**6.** Write down the full solution for the AC RC circuit, with the initial condition that at $t = 0$, the initial charge on the capacitor was 0.

**Answer:** Combining the particular and "homogeneous" (or transient) solutions,

$$Q = \frac{CV_0}{1 + iRC\omega}\, e^{i\omega t} + Q_0\, e^{-t/RC}$$

At $t = 0$, $0 = A + Q_0$, therefore $Q_0 = -A$, and

$$Q = \frac{CV_0}{1 + iRC\omega} \left(e^{i\omega t} - e^{-t/RC}\right)$$

The exponential decay part typically quickly goes away, hence it's called "transient."