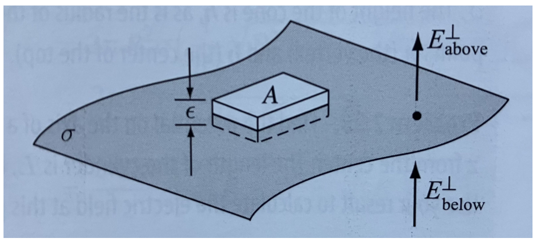
PHYS 191 Exam 2

1. (40 points) Here are a couple of figures from an E&M textbook, concerning the boundary conditions for an electric field around a surface with constant charge density σ and no other charges in the vicinity.

(a) The first figure gives a Gaussian surface with a small area A and thickness ϵ. Using Gauss's Law and taking $\epsilon \to 0$, find the discontinuity in the electric field component perpendicular to the surface, $E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}}$.

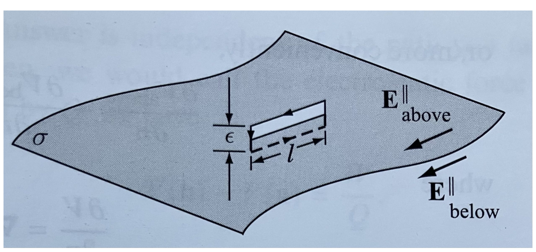


Answer: Since $\epsilon \to 0$, the contribution to the closed surface integral from the sides of the Gaussian surface will go to zero. The contributions from the areas above and below the surface result in

$$\oint \vec{E} \cdot d\vec{A} = E^{\perp}_{\text{above}} A - E^{\perp}_{\text{below}} A = \frac{1}{\epsilon_0} \sigma A \quad \Rightarrow \quad E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{\sigma}{\epsilon_0}$$

(b) The second figure gives a loop with a small length l and end width ϵ. As $\epsilon \to 0$, find the discontinuity in the electric field component parallel to the surface, $E^{\parallel}_{\text{above}} - E^{\parallel}_{\text{below}}$.

Hint: In electro*static* conditions, what is $\oint \vec{E} \cdot d\vec{l}$?



Answer: Electrostatics mean a closed loop integral $\oint \vec{E} \cdot d\vec{l} = 0$. With $\epsilon \to 0$, the contribution to the closed line integral from the sides of the small loop will go to zero. The contributions from the line segments above and below the surface result in

$$\oint \vec{E} \cdot d\vec{l} = E^{\parallel}_{\text{above}} l - E^{\parallel}_{\text{below}} l = 0 \quad \Rightarrow \quad E^{\parallel}_{\text{above}} - E^{\parallel}_{\text{below}} = 0$$

2. (60 points) You have a disc with radius R and uniform surface charge density σ rotating around its center with constant angular velocity $\vec{\omega} = \omega\hat{\mathbf{z}}$. (The axis of rotation is the z-axis.)

(a) Take a ring of charge with radius r and thickness dr. What is the total charge dq on this ring of charge? Since as the ring spins, that charge dq flows by every period T, the current on that ring is $dI = dq/T$. Remembering that for circular motion $\omega = 2\pi/T$, find the current dI flowing through that ring.

Answer: The area of the ring is $(2\pi r)dr$, and therefore $dq = \sigma(2\pi r)dr$, resulting in

$$dI = \frac{2\pi\sigma r\, dr}{2\pi/\omega} = \omega\sigma r\, dr$$

(b) Find the $\vec{B}$ field produced by this rotating disc at a location along the z-axis.

Hint: For your integral, you will need a $u^2 = r^2 + z^2$ substitution.

Answer: From your class notes, you can find the magnetic field along the z-axis produced by a loop of radius R with current I. Due to symmetry, the only non-zero component is

$$B_z = \frac{\mu_0 I R^2}{2\left(R^2 + z^2\right)^{3/2}}$$

Therefore, for a ring of charge with width dr we are considering,

$$dB_z = \frac{\mu_0 dI\, r^2}{2\left(r^2 + z^2\right)^{3/2}} = dr\, \frac{\mu_0\omega\sigma}{2} \frac{r^3}{\left(r^2 + z^2\right)^{3/2}}$$

Integrating this,

$$\begin{aligned} B_z &= \int_{\text{disc}} dB_z = \frac{\mu_0\omega\sigma}{2}\int_0^R dr\, \frac{r^3}{\left(r^2+z^2\right)^{3/2}} = \frac{\mu_0\omega\sigma}{2}\int_{|z|}^{\sqrt{R^2+z^2}} du\, \frac{u(u^2 - z^2)}{u^3} \\ &= \frac{\mu_0\omega\sigma}{2}\left[u + \frac{z^2}{u}\right]_{|z|}^{\sqrt{R^2+z^2}} = \frac{\mu_0\omega\sigma}{2}\left(\frac{2z^2 + R^2}{\sqrt{z^2 + R^2}} - 2|z|\right) \end{aligned}$$