Phys 191 Activity 8: Magnetic Fields

Magnetism is *analogous*, but not identical, to electricity. Each relation for electric fields and charges has a similar one for magnetic fields and currents (moving charges).

Coulomb's Law becomes the Biot-Savart law:

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\int dq\,\frac{\hat{\mathbf{r}}}{r^2} \qquad \longrightarrow \qquad \vec{B} = \frac{\mu_0}{4\pi}\int \frac{I\,d\vec{l}\times\hat{\mathbf{r}}}{r^2}$$

The sources for magnetic fields are not points, but lines of current. So dq becomes $I\,dl$. The directionality is also different, because magnetic fields wrap around currents instead of pointing radially out. That is represented by $d\vec{l}\times\hat{\mathbf{r}}$—a cross product.

1. Find the magnetic field of a circular loop with current I and radius R along its axis of symmetry. The problem is similar but not identical to finding the electric field of a circular charge distribution.

Answer: Choose a point on the ring at $y = 0$ and $x = R$. If the current is going counterclockwise, $d\vec{l} = (R\,d\phi)\hat{\mathbf{y}}$. The unit vector directed toward our point on the z-axis is $\hat{\mathbf{r}} = -R/r\,\hat{\mathbf{x}} + z/r\,\hat{\mathbf{z}}$, with $r = \sqrt{R^2+z^2}$. Therefore,

$$d\vec{l}\times\hat{\mathbf{r}} = (R\,d\phi)\left(-\frac{R}{r}\,\hat{\mathbf{y}}\times\hat{\mathbf{x}} + \frac{z}{r}\,\hat{\mathbf{y}}\times\hat{\mathbf{z}}\right) = d\phi\,\frac{R}{r}\,(R\,\hat{\mathbf{z}} + z\,\hat{\mathbf{x}})$$

Due to symmetry, only $B_z \neq 0$ along the z-axis. To see that, you can also think of a bit of current exactly opposite on the ring; it has an equal and opposite $d\vec{l}$, and so its contribution to B_x will cancel out the one we just calculated. The total magnetic field is

$$\vec{B} = \oint_{\text{ring}} d\vec{B} = \frac{\mu_0 I}{4\pi}\int_0^{2\pi} d\phi\,\frac{R^2}{r^3}\,\hat{\mathbf{z}} = \frac{\mu_0 I R^2}{2\left(R^2+z^2\right)^{3/2}}\,\hat{\mathbf{z}}$$

When $z = 0$, this gives $B_z = \mu_0 I/2R$, which is correct.

2. Gauss's Law for electric fields relied on the properties of electric field lines going radially outward from charges; hence calculating the electric flux was a useful thing to do. A hint was that the point charge field fell off like the surface area of a sphere ($4\pi r^2$).

The magnetic field of a long wire with current I loops around the wire, and has magnitude $B = \mu_0 I/2\pi r$. What sort of integral should we be doing instead? Can you guess "Ampère's Law" which is the magnetic analog of Gauss's Law?

Answer: $2\pi r$ is the circumference of a circle—the circle that a magnetic field line makes. This suggests a line integral rather than a flux integral. Instead of the total charge inside a closed surface, we need the total current going through a closed loop. Ampère's Law is, in fact,

$$\oint_{\text{loop}} d\vec{l}\cdot\vec{B} = \mu_0 I_{\text{through}}$$

With an infinitely long straight wire, $\vec{B} \parallel d\vec{l}$, and so,

$$\oint d\vec{l} \cdot \vec{B} = \oint dl\, B = 2\pi r\, B = \mu_0 I \quad \Rightarrow \quad B = \frac{\mu_0 I}{2\pi r}$$

3. The magnetic force exerted on a charge q moving through a magnetic field $\vec{B}$ is $q\,\vec{v} \times \vec{B}$, where $\vec{v}$ is the velocity vector of the charge and $\times$ is the vector cross product.

How much work dW is done by the magnetic force during an interval dt? (Then $d\vec{r} = \vec{v}\, dt$.)

Can you define a magnetic potential energy U_B corresponding to the magnetic field, like we can the electric field?

Answer: The magnetic force is always perpendicular to the velocity, and therefore always perpendicular to any distance traveled due to the magnetic force. Magnetic forces do no work!

$$dW = \vec{F} \cdot d\vec{r} = q(\vec{v} \times \vec{B}) \cdot \vec{v}\, dt = 0$$

No work done means there is no potential energy associated with magnetism, and there is no such thing as magnetic potential analogous to electric potential. In more advanced coursework, you will encounter a "vector potential" $\vec{A}$ such that $\vec{\nabla} \times \vec{A} = \vec{B}$, but that is a different animal.