# PHYS 191 Activity 9: Rotations and Lorentz boosts

Special relativity mixes space and time. In fact, relativity is related to rotations in space, which mixes spatial coordinates.

**1.** We can express rotation by $\theta$ in 2D as $x' = x\cos\theta + y\sin\theta$ and $y' = y\cos\theta - x\sin\theta$. Equivalently, in matrix notation,

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Draw $x$ and $y$ coordinate axes, and also $x'$ and $y'$ axes that share the same origin but are rotated by $\theta$. Put in a point with coordinates $x$ and $y$. Find its $x'$ and $y'$ coordinates, verifying the above.

**Answer:** Lots of triangles.

**2.** Let's say we have two points, with coordinates $x_1$, $y_1$ and $x_2$, $y_2$. The distance between these points is invariant under rotations:

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 = (x_1' - x_2')^2 + (y_1' - y_2')^2$$

Show that this is so—that rotations preserve lengths.

**Answer:** Multiply things out:

$$\begin{aligned}
(x_1' - x_2')^2 + (y_1' - y_2')^2 &= (x_1\cos\theta + y_1\sin\theta - x_2\cos\theta - y_2\sin\theta)^2 + \\
&\quad (y_1\cos\theta - x_1\sin\theta - y_2\cos\theta + x_2\sin\theta)^2 \\
&= (x_1 - x_2)^2\cos^2\theta + 2(x_1 - x_2)(y_1 - y_2)\sin\theta\cos\theta + \\
&\quad (y_1 - y_2)^2\sin^2\theta + (x_1 - x_2)^2\sin^2\theta - \\
&\quad 2(x_1 - x_2)(y_1 - y_2)\sin\theta\cos\theta + (y_1 - y_2)^2\cos^2\theta \\
&= \left[(x_1 - x_2)^2 + (y_1 - y_2)^2\right](\sin^2\theta + \cos^2\theta) \\
&= (x_1 - x_2)^2 + (y_1 - y_2)^2
\end{aligned}$$

**3.** A Lorentz boost relates two non-accelerating reference frames with a relative speed of $v = \beta c$. In 1+1D, this is expressed by a Lorentz transformation: $ct' = \gamma ct - \beta\gamma x$ and $x' = -\beta\gamma ct + \gamma x$.

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix} \quad \text{with} \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

Show that if $\gamma = \cosh\eta$, then $\beta\gamma = \sinh\eta$, so that

$$\begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} = \begin{pmatrix} \cosh\eta & -\sinh\eta \\ -\sinh\eta & \cosh\eta \end{pmatrix} \qquad \text{A hyperbolic rotation!}$$

Here, $\cosh\eta = \frac{1}{2}(e^{\eta} + e^{-\eta})$ and $\sinh\eta = \frac{1}{2}(e^{\eta} - e^{-\eta})$, just like $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ and $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$. Use $\cosh^2\eta - \sinh^2\eta = 1$.

**Answer:** Since $\cosh^2\eta = \gamma^2$ and $\sinh^2\eta = \cosh^2\eta - 1$,

$$\beta^2\gamma^2 = \frac{\beta^2}{1-\beta^2} = \frac{1}{1-\beta^2} - 1 = \gamma^2 - 1 = \cosh^2\eta - 1 = \sinh^2\eta$$

**4.** Let's say we have two events, with coordinates $ct_1$, $x_1$ and $ct_2$, $x_2$. The distance between these events is *not* invariant under Lorentz transformations:

$$(ct_1 - ct_2)^2 + (x_1 - x_2)^2 \neq (ct'_1 - ct'_2)^2 + (x'_1 - x'_2)^2$$

Show that this is so. What *is* invariant?

**Answer:** Multiplying this out,

$$\begin{aligned}
(ct'_1 - ct'_2)^2 + (x'_1 - x'_2)^2 &= (ct_1\cosh\eta - x_1\sinh\eta - ct_2\cosh\eta + x_2\sinh\eta)^2 + \\
&\quad (x_1\cosh\eta - ct_1\sinh\eta - x_2\cosh\eta + ct_2\sinh\eta)^2 \\
&= (ct_1 - ct_2)^2\cosh^2\eta - 2(ct_1 - ct_2)(x_1 - x_2)\sinh\eta\cosh\eta + \\
&\quad (x_1 - x_2)^2\sinh^2\eta + (ct_1 - ct_2)^2\sinh^2\eta - \\
&\quad 2(ct_1 - ct_2)(x_1 - x_2)\sinh\eta\cosh\eta + (x_1 - x_2)^2\cosh^2\eta \\
&= \left[(ct_1 - ct_2)^2 + (x_1 - x_2)^2\right](\sinh^2\eta + \cosh^2\eta) - \\
&\quad 4(ct_1 - ct_2)(x_1 - x_2)\sinh\eta\cosh\eta
\end{aligned}$$

This doesn't usefully simplify. But

$$\begin{aligned}
(ct'_1 - ct'_2)^2 - (x'_1 - x'_2)^2 &= (ct_1\cosh\eta - x_1\sinh\eta - ct_2\cosh\eta + x_2\sinh\eta)^2 - \\
&\quad (x_1\cosh\eta - ct_1\sinh\eta - x_2\cosh\eta + ct_2\sinh\eta)^2 \\
&= (ct_1 - ct_2)^2\cosh^2\eta - 2(ct_1 - ct_2)(x_1 - x_2)\sinh\eta\cosh\eta + \\
&\quad (x_1 - x_2)^2\sinh^2\eta - (ct_1 - ct_2)^2\sinh^2\eta + \\
&\quad 2(ct_1 - ct_2)(x_1 - x_2)\sinh\eta\cosh\eta - (x_1 - x_2)^2\cosh^2\eta \\
&= \left[(ct_1 - ct_2)^2 - (x_1 - x_2)^2\right](\cosh^2\eta - \sinh^2\eta) \\
&= (ct_1 - ct_2)^2 - (x_1 - x_2)^2
\end{aligned}$$

This is invariant! It's a sort of relativistic invariant length, which is zero for photons.