Solutions to Assignment 7; Phys 185

1. (30 points) You're in a spaceship in a circular orbit around a planet. Your distance to the planet's center is r. The captain of your spaceship decides to take a closer look, and so she maneuvers your ship to a distance of $r/2$ to the planet's center. (It's still well clear of the planet's surface.) During the movement to get closer, the only forces applied on the spaceship are radial in direction, with zero tangential component. (Radial means straight inward or outward; tangential is tangent to a circle centered on the planet.) The maneuvers use a negligible amount of fuel, so that m_s, the mass of the ship, remains constant. And when you reach $r/2$, the spaceship has a radial velocity component of zero, just as it was at r.

You have three possibilities for when you reach $r/2$:

Just right. You have a circular orbit with radius $r/2$, without having to do anything else.

Too fast. You need to slow down the spaceship to remain in a circular orbit with radius $r/2$.

Too slow. You need to speed up the spaceship to remain in a circular orbit with radius $r/2$.

Which one is correct? Produce a calculation that shows why.

Answer: $\sum \vec{F} = m_s \vec{a}$ with gravity and uniform circular motion gives

$$G\frac{m_p m_s}{r^2} = m_s \omega_i^2 r \qquad \Rightarrow \qquad \omega_i = \sqrt{\frac{Gm_p}{r^3}}$$

When changing orbit, all the forces on the ship are radial: directed toward the axis of rotation. Therefore none of these forces produce any torque. Gravity is radial, which also does not produce torque. The angular momentum of the spaceship will be conserved. $L = I\omega$, and since the size of the starship is much smaller than r, we can approximate it as a point mass, with $I = m_s r^2$. Setting the initial and final angular momenta equal,

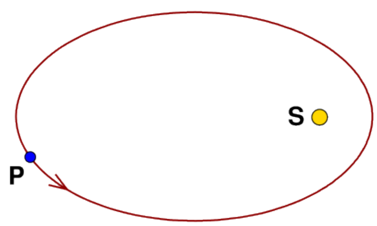
$$\left(m_s r^2\right)\omega_i = \left[m_s\left(\frac{r}{2}\right)^2\right]\omega_f \qquad \Rightarrow \qquad \omega_f = 4\omega_i$$

Now, for uniform circular motion with radius $r/2$, the angular velocity must be

$$\omega_f = \sqrt{\frac{Gm_p}{(r/2)^3}} = 2^{3/2}\sqrt{\frac{Gm_p}{r^3}} = 2^{3/2}\omega_i$$

Since $4 = 2^2 > 2^{3/2}$, the spaceship will be going too fast to have a circular orbit. Doing nothing, it would settle into an elliptical orbit. To have a circle, it must slow down.

2. (30 points) A planet (P) with mass m revolves around a star (S) with mass M, where $M \gg m$, so that the center of mass of the system is almost at the center of the star itself. The orbit is an ellipse, and the star is not at the center of the ellipse, but at one of the focal points. Call the minimum distance between the star and the planet a, and the maximum distance b. The speed of the planet when it is closest to the star is v_a, when it is farthest it's v_b.



(a) Briefly explain why you can use energy conservation in this case, and write down the energy conservation equation that relates the total energy when the planet is closest to the total energy when it is farthest.

Answer: In the vacuum of outer space, there are no dissipative forces that create losses that are difficult to account for. Therefore,

$$\tfrac{1}{2}mv_a^2 - G\frac{mM}{a} = \tfrac{1}{2}mv_b^2 - G\frac{mM}{b}$$

(b) You can also use angular momentum, with $\omega_a = v_a/a$ and $\omega_b = v_b/b$. Briefly explain why angular momentum is conserved during the planet's orbit, and write down the equation that relates the angular momentum when the planet is closest to when it is farthest.

Answer: Angular momentum is conserved because the only force on the planet is the gravity of the star, which is always directed toward the star, creating no torque around the star.

a and b are much larger than the size of the planet, so its moment of inertia is that of a point particle. Therefore,

$$\left(ma^2\right)\frac{v_a}{a} = \left(mb^2\right)\frac{v_b}{b} \qquad \Rightarrow \qquad av_a = bv_b$$

(c) Check if the linear momentum is conserved by looking at the linear momentum of the planet when it is closest and farthest. Explain your result.

Answer: Linear momentum is clearly not conserved: $\vec{p}_a$ and $\vec{p}_b$ are in opposite directions. The planet does have a nonzero total external force on it: the gravity of the star. Note: the *vector* nature of momentum is important if you want to have an argument that is valid for circular orbits where $a = b$. Just pointing out differing magnitudes is not good enough.

(d) Combine your valid conservation equations to eliminate v_b and find an equation for v_a in terms of G, M, a, and b. To simplify your algebra, you can use

$$\frac{\left(\frac{1}{a} - \frac{1}{b}\right)}{1 - \left(\frac{a}{b}\right)^2} = \frac{b}{a(a+b)}$$

Answer: From angular momentum conservation, $v_b = \frac{a}{b} v_a$. Putting that into the energy conservation equation,

$$\tfrac{1}{2} m v_a^2 - \tfrac{1}{2} m \left(\frac{a}{b}\right)^2 v_a^2 = GmM \left(\frac{1}{a} - \frac{1}{b}\right) \qquad \Rightarrow \qquad v_a^2 = 2GM \frac{\left(\frac{1}{a} - \frac{1}{b}\right)}{1 - \left(\frac{a}{b}\right)^2}$$

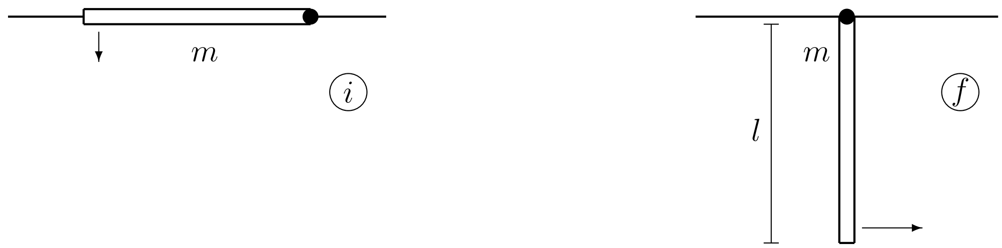
Simplifying,

$$v_a = \sqrt{2GM \frac{b}{a(a+b)}}$$

(e) Use your result to explain how astronomers can obtain the mass of a star if they can observe the details of the orbit of a smaller mass around this star.

Answer: The details of the orbit include a, b, and v_a. If an astronomer obtains these quantities, she can calculate $M = a(a+b)v_a^2/2Gb$. Note that the result does not depend on m; everything assumed that $M \gg m$.

3. (40 points) You have a uniform thin rod attached to the ceiling at one end. Starting from the rod being up against the ceiling, at rest, you let the rod go, and it swings down. The mass of the rod is m, and its length is l. The effects of the drag force on the rod, and the friction at the ceiling, are negligible. In the following questions, when a variable has a subscript f, it refers to what is happening at the bottom of the arc of the rod's swing, when it is positioned completely vertically.



(a) The moment of inertia of a rod rotating around its center of mass is $ml^2/12$. What is its moment of inertia when the axis of rotation is the point of attachment to the ceiling?

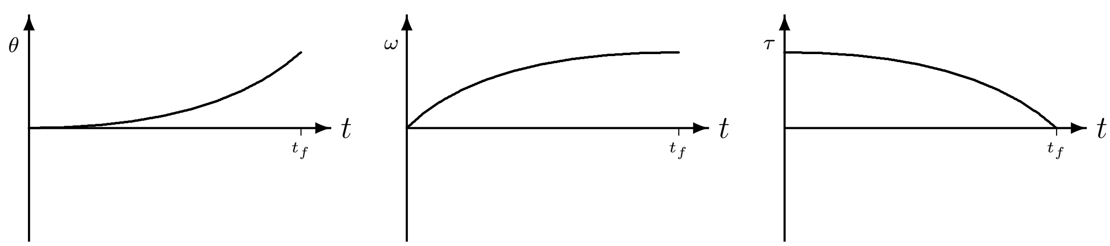
Answer: The parallel axis theorem gives $I = ml^2/12 + m(l/2)^2 = ml^2/3$.

(b) Say your y-axis is pointing upward, and the ceiling height is $y = 0$, so that the initial gravitational potential energy is $U_i = 0$. Circle the potential energy of the rod in position f, as it is vertical. Then provide a reason for your choice.

$$-mgl \qquad -\tfrac{1}{2}mgl \qquad -\tfrac{1}{3}mgl \qquad -\tfrac{1}{6}mgl \qquad -\tfrac{1}{12}mgl$$

Answer: The center of mass of the rod is at its middle, a distance $l/2$ lower than the ceiling. Therefore, $U_f = -\frac{1}{2}mgl$.

(c) Make qualitative graphs of θ, the angle of the rod with the ceiling; ω, its angular velocity; and τ, the total torque on the rod. Give the values of θ and τ when $t = t_i = 0$ and $t = t_f$, the time when the rod is vertical. Briefly explain your reasoning.



Answer: The angle with the ceiling starts from 0 and goes to $\pi/2$. Because of the counterclockwise rotation, ω is positive, and increasingly so; $\omega = \frac{d}{dt}\theta$. And then, the torque $\tau \propto \alpha = \frac{d}{dt}\omega$. τ is due to the $\perp$ component of the rod's weight, which decreases in magnitude as the rod falls. At the time of release, $\tau = mgl/2$; at t_f, $\tau = 0$.

(d) Can you use linear momentum conservation to find out the angular velocity ω_f? If so, calculate ω_f. If not, explain why.

Answer: No. The rod has a non-zero external total force on it: its weight and the forces at the point of rotation (normal force) do not cancel out. You can see this because the center of mass of the rod accelerates; therefore, $\sum \vec{F} \neq 0$.

(e) Can you use energy conservation to find ω_f? If so, calculate ω_f. If not, explain why.

Answer: Yes. With negligible drag and friction, $E_{\text{loss}} = 0$. Therefore,

$$0 + 0 = \tfrac{1}{2}(\tfrac{1}{3}ml^2)\omega_f^2 - \tfrac{1}{2}mgl \qquad \Rightarrow \qquad \omega_f = \sqrt{\frac{3g}{l}}$$

(f) Can you use angular momentum conservation to find ω_f? If so, calculate ω_f. If not, explain why.

Answer: No. The rod has a non-zero external total torque on it due to its weight. The forces at the point of contact at the ceiling act on the axis of rotation, and therefore produce no torque.

Extra Problems (not graded)

4. (0 points) You have a horizontal, disc-shaped platform that rotates around its center. Its mass is M and radius is R. There is no friction at the axis of rotation. On the platform, at radius $r = R/2$, there is a kid with mass $m = M/40$. They both are initially at rest.

(a) The kid steps forward in such a fashion that her r remains constant, propelled by a force $\vec{F}$ due to the friction between the platform and her shoes. As she moves, what is the ratio of the angular acceleration of the kid to the angular acceleration of the platform?

Answer: The torque on the kid is $\tau_k = F_\perp r = F_\perp R/2$. This is also the only torque on the kid, so that

$$\tau_k = F_\perp R/2 = I_k \alpha_k = mr^2 \alpha_k = \frac{M}{40}\left(\frac{R}{2}\right)^2 \alpha_k \qquad \Rightarrow \qquad \alpha_k = \frac{80\,F_\perp}{MR}$$

Note that the moment of inertia of the kid is that of a point particle, since she is much smaller than r.

The force on the platform at that point is $-\vec{F}$, since the forces are action-reaction pairs. So a similar calculation gives

$$\tau_p = -F_\perp R/2 = I_p \alpha_p = \frac{1}{2} M R^2 \alpha_p \quad \Rightarrow \quad \alpha_p = -\frac{F_\perp}{MR}$$

Therefore the ratio is

$$\frac{\alpha_k}{\alpha_p} = -\frac{I_p}{I_k} = -80$$

(b) The kid keeps moving in a manner such that her r is constant. What is the ratio of the angular velocity of the kid to the angular velocity of the platform at any instant?

Answer: There are two ways to do this. First, since there are no *external* torques on the system consisting of the kid and platform, their total angular momentum will be conserved. Since everything starts from rest, $L_{\text{total}} = 0$. When the kid is moving,

$$L_{\text{total}} = I_k \omega_k + I_p \omega_p = 0 \quad \Rightarrow \quad \frac{\omega_k}{\omega_p} = -\frac{I_p}{I_k} = -80$$

Second, in general,

$$\omega = \omega_0 + \text{Area under the } \alpha(t) \text{ curve}$$

In our case, $\omega_0 = 0$. Therefore,

$$\frac{\omega_k}{\omega_p} = \frac{\text{Area under the } \alpha_k(t) \text{ curve}}{\text{Area under the } \alpha_p(t) \text{ curve}} = \frac{\alpha_k}{\alpha_p} = -80$$

Since the fraction α_k/α_p remains constant, the area ratio will also be that ratio.

(c) What is the ratio of the total angle covered by the kid since she started moving to the angle rotated through by the platform?

Answer: The second form of reasoning applies here. In general,

$$\theta = \theta_0 + \text{Area under the } \omega(t) \text{ curve}$$

In our case, $\theta_0 = 0$. Therefore,

$$\frac{\theta_k}{\theta_p} = \frac{\text{Area under the } \omega_k(t) \text{ curve}}{\text{Area under the } \omega_p(t) \text{ curve}} = \frac{\omega_k}{\omega_p} = -80$$

Since the fraction ω_k/ω_p remains constant, the area ratio will also be that ratio.

5. (0 points) Astronomers observe a new comet approaching the sun. They obtain the location of the comet relative to the sun, and the velocity of the comet. But they don't have long-term data to directly tell whether the trajectory of the comet is an ellipse or a hyperbola. Still, astronomers can figure it out. After all, an elliptical orbit means that the comet is gravitationally bound to the sun: it can never escape to an infinite distance. But a hyperbolic trajectory extends to infinity: the comet is unbound and must escape the Sun's gravity.

Chose, from the following options, the test that astronomers can apply, using their position and velocity observations, that distinguishes between a bound and an unbound comet. If an option is incorrect, briefly explain why. If it is correct, give the exact inequality they will apply, using the velocity and distance to the sun of the comet, and if needed, data about other objects in the solar system. State whether the inequality indicates a bound or unbound comet.

(a) $L_{\text{comet}} > L_{\text{Earth}}$. (Compare angular momenta.)

Answer: This is a pointless comparison: L_{Earth} is irrelevant to the comet. Angular momentum conservation applies to the comet, yes, but the Earth doesn't come into that.

(b) $E_{\text{comet}} > 0$. (The total orbital energy of the comet is positive.)

Answer: This will work. The total energy of the comet is conserved, so

$$\frac{1}{2} m_{\text{comet}} v_{\text{comet}}^2 - G \frac{m_{\text{comet}} m_{\text{Sun}}}{r_{\text{comet}}} < 0$$

will mean that the comet is bound. The reason is the same one in your Assignment 6, problem 3.

Since the mass of the comet is unknown, but the mass of the Sun is, cancel out the comet mass—the inequality the astronomers can check is

$$\frac{1}{2} v_{\text{comet}}^2 - G \frac{m_{\text{Sun}}}{r_{\text{comet}}} < 0$$

(c) $I_{\text{comet}} > I_{\text{Sun}}$. (Compare moments of inertia.)

Answer: Again, this is completely irrelevant. The moment of inertia says nothing about whether the comet has enough energy to escape the Sun's gravity.

(d) $\Delta \vec{p}_{\text{comet}} > \vec{F}_{\text{comet}}$. (Compare the momentum change to the gravity on the comet.)

Answer: This makes no sense. Even the units on the right and left sides don't match.

(e) $\sum \tau_{\text{comet}} > 0$. (The total torque on the comet is positive.)

Answer: The gravitational force felt by the comet will be toward the sun, which is also the direction of the axis of rotation. This means that $\sum \tau_{\text{comet}} = 0$, regardless of whether the comet can escape.