Solutions 10; Phys 191 Name ______________

When you include time-dependence, quantum wave functions are solutions to the Schrödinger equation:

$$i\hbar\frac{\partial\Psi}{\partial t} = \hat{H}\Psi$$

The *Hamiltonian* operator $\hat{H}$ is the total energy. With a 1D particle in a box, that's just the kinetic energy: $\hat{H} = \hat{K}$. All quantum states can be expressed as a superposition of the energy eigenfunctions we have found before. Say your wave function combines the first and second energy levels:

$$\Psi(x,t) = N\left[\psi_1(x)e^{-i\omega_1 t} + i\psi_2(x)e^{-i\omega_2 t}\right]$$

Here, ψ_n refers to the nth energy eigenfunction and $\omega_n = K_n/\hbar$. N is a normalization constant.

1. (30 points) Show that Ψ satisfies the Schrödinger equation. (Plug it in.)

Answer:

$$\begin{aligned} i\hbar\frac{\partial\Psi}{\partial t} &= iN\hbar\frac{\partial}{\partial t}\left[\psi_1(x)e^{-i\omega_1 t} + i\psi_2(x)e^{-i\omega_2 t}\right] = N\hbar\left[\omega_1\psi_1(x)e^{-i\omega_1 t} + i\omega_2\psi_2(x)e^{-i\omega_2 t}\right] \\ &= N\left[K_1\psi_1(x)e^{-i\omega_1 t} + iK_2\psi_2(x)e^{-i\omega_2 t}\right] \end{aligned}$$

Notice that $\frac{\partial^2}{\partial x^2}\psi_n = -(n\pi/L)^2\psi_n = -2mK_n\psi_n/\hbar^2$. Therefore,

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi = N\left[K_1\psi_1(x)e^{-i\omega_1 t} + iK_2\psi_2(x)e^{-i\omega_2 t}\right]$$

These are the same, so Schrödinger's equation is satisfied.

2. (20 points) You know that the ψ_n are normalized:

$$\int_0^L dx\,\psi_1^*\psi_1 = \int_0^L dx\,\psi_2^*\psi_2 = 1$$

Show that ψ_1 and ψ_2 are *orthogonal*:

$$\int_0^L dx\,\psi_1^*\psi_2 = 0$$

Answer:

$$\int_0^L dx\,\psi_1^*\psi_2 = \frac{2}{L}\int_0^L dx\,\sin\left(\frac{\pi}{L}x\right)\sin\left(\frac{2\pi}{L}x\right) = 0$$

3. (50 points) Find N, the normalization constant.

Answer: Normalization means that

$$\int_0^L dx\, \Psi^*\Psi = N^*N \int_0^L dx\, \left[\psi_1^*(x)e^{i\omega_1 t} - i\psi_2^*(x)e^{i\omega_2 t}\right]\left[\psi_1(x)e^{-i\omega_1 t} + i\psi_2(x)e^{-i\omega_2 t}\right] = 1$$

There are four integrals to do. We can use normalization or orthogonality for all.

$$\int_0^L dx\, \psi_1^*\psi_2\, e^{i\omega_1 t}e^{-i\omega_1 t} = \int_0^L dx\, \psi_1^*\psi_2 = 1$$

$$i\int_0^L dx\, \psi_1^* e^{i\omega_1 t}\psi_2 e^{-i\omega_2 t} = ie^{-i(\omega_1-\omega_2)t}\int_0^L dx\, \psi_1^*\psi_2 = 0$$

$$-i\int_0^L dx\, \psi_2^* e^{i\omega_2 t}\psi_1 e^{-i\omega_1 t} = ie^{-i(\omega_2-\omega_1)t}\int_0^L dx\, \psi_2^*\psi_1 = 0$$

$$-i^2\int_0^L dx\, \psi_2^*\psi_2\, e^{i\omega_2 t}e^{-i\omega_2 t} = \int_0^L dx\, \psi_2^*\psi_2 = 1$$

Therefore

$$\int_0^L dx\, \Psi^*\Psi = N^*N(1+1) = 1$$

and

$$N = \frac{1}{\sqrt{2}}$$