PHYS 191 Solutions 10: A bit of quantum

A quantum wave function ψ is a complex-valued function. And physical observables are not ordinary numbers; instead, they're associated with *operators*. For example, for a 1D particle, the momentum operator is $\hat{p} = -i\hbar\frac{\partial}{\partial x}$.

1. The spectrum of an observable—the possible values you can obtain when you make a measurement of that observable, are the *eigenvalues* you get from solving the eigenvalue equation. For momentum, that is

$$\hat{p}\psi_p = p\psi_p \qquad \text{or} \qquad -i\hbar\frac{\partial\psi_p}{\partial x} = p\psi_p$$

the p in this equation is an ordinary real number; these are the momentum eigenvalues. The ψ_p are the *eigenfunctions* with eigenvalues p. Find the possible values of p and the corresponding $\psi_p(x)$.

Answer: You should know by now that the solution is an exponential: $\psi_p = Ae^{ikx}$.

$$-i\hbar\frac{\partial Ae^{ikx}}{\partial x} = (\hbar k)Ae^{ikx}$$

The eigenvalues are $p = \hbar k = h/\lambda$. All $-\infty < p < \infty$ are allowed.

2. Classically, the nonrelativistic kinetic energy $K = p^2/2m$. The quantum kinetic energy operator is $\hat{K} = \hat{p}^2/2m$. Write $\hat{K}$ in terms of derivatives.

Answer: $\hat{p}^2$ means operate by $\hat{p}$ twice: a derivative of a derivative. Therefore,

$$\hat{K} = \frac{1}{2m}(-i\hbar)(-i\hbar)\frac{\partial}{\partial x}\frac{\partial}{\partial x} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$$

3. A free particle only has kinetic energy. Find the energy eigenvalues K and energy eigenfunctions ψ_K for a free particle.

Answer: The eigenvalue equation is

$$\frac{\hbar^2}{2m}\frac{\partial^2\psi_K}{\partial x^2} = K\psi_K$$

Again, you should look for exponentials: $\psi_K = Ae^{ikx}$.

$$\frac{\hbar^2}{2m}A\frac{\partial^2 e^{ikx}}{\partial x^2} = \frac{\hbar^2 k^2}{2m}Ae^{ikx}$$

In other words, $K = \hbar^2 k^2/2m$. But notice that both $\pm k$ give the same energy K. So the energy eigenfunctions are a combination of the two:

$$\psi_K = Ae^{ikx} + Be^{-ikx}$$

Alternatively, this is

$$\psi_K = A(\cos kx + i\sin kx) + B(\cos kx - i\sin kx) = \alpha\cos kx + \beta\sin kx$$

with $\alpha = A + B$ and $\beta = i(A - B)$.

4. Is every momentum eigenfunction also a kinetic energy eigenfunction? Is every kinetic energy eigenfunction also a momentum eigenfunction?

Answer: Every momentum eigenfunction with eigenvalue $p = \hbar k$ is also a kinetic energy eigenfunction with $K = \hbar^2 k^2/2m$. But the reverse is not true unless $A = 0$ or $B = 0$.

5. The 1D particle in a box is a free particle, but the wave function is also constrained to be zero at the box boundaries: $\psi(0) = 0$ and $\psi(L) = 0$. Find the energy eigenvalues and eigenfunctions for a 1D particle in a box.

Answer: First, apply $\psi(0) = 0$:

$$Ae^0 + Be^{-0} = 0 \qquad \Rightarrow \qquad A = -B$$

Then, $\psi(L) = 0$:

$$A(e^{ikL} - e^{-ikL}) = 0 \quad \text{or} \quad \beta\sin kL = 0 \qquad \Rightarrow \qquad kL = n\pi,\ n = 1, 2, 3, \ldots$$

This should be familiar from standing waves. These *are* standing waves!

The energy eigenvalues and eigenfunctions are

$$K_n = \frac{\hbar^2\left(\frac{n\pi}{L}\right)^2}{2m} \quad \text{and} \quad \psi_n = \beta\sin\left(\frac{n\pi}{L}x\right)$$

The energies are *quantized*; we have energy *levels*.

6. The 1D particle in a box wave functions can be *normalized*:

$$\int_0^L dx\,\psi^*\psi = 1$$

Find the normalized energy eigenfunctions for a 1D particle in a box

Answer: Normalize away:

$$\int_0^L dx\,\left[\beta^*\sin\left(\frac{n\pi}{L}x\right)\right]\left[\beta\sin\left(\frac{n\pi}{L}x\right)\right] = |\beta|^2\int_0^L dx\,\sin^2\left(\frac{n\pi}{L}x\right) = \frac{|\beta|^2 L}{2} = 1$$

Therefore $|\beta| = \sqrt{2/L}$, and

$$\psi_K = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi}{L}x\right)$$