PHYS 191 Solutions 11: Position expectation value

First, a couple of integrals we will need, and a trick to do them. Let's call

$$I(\alpha) = \int_0^L dx\, e^{i\alpha x/L} = \left.\frac{e^{i\alpha x/L}}{i\alpha/L}\right|_0^L = \frac{L}{i\alpha}\left(e^{i\alpha} - 1\right)$$

In that case,

$$\begin{aligned} \frac{L}{i}\frac{\partial I}{\partial \alpha} &= \frac{L}{i}\int_0^L dx\, \frac{\partial e^{i\alpha x/L}}{\partial \alpha} = \frac{L}{i}\int_0^L dx\, \frac{ix}{L} e^{i\alpha x/L} = \int_0^L dx\, x\, e^{i\alpha x/L} \\ &= -L^2\left[-\alpha^{-2}(e^{i\alpha} - 1) + \alpha^{-1} i e^{i\alpha}\right] = \frac{L^2}{\alpha}\left[\frac{1}{\alpha}(e^{i\alpha} - 1) - ie^{i\alpha}\right] \end{aligned}$$

With $\alpha = n\pi$, where $n \neq 0$ is an integer, $e^{in\pi} = (-1)^n$. Therefore

$$\int_0^L dx\, e^{i\alpha x/L} = \frac{L}{in\pi}\left(e^{in\pi} - 1\right) = \frac{L}{in\pi}\left((-1)^n - 1\right)$$

$$\int_0^L dx\, x\, e^{i2n\pi x/L} = \frac{L^2}{n\pi}\left[\frac{1}{n\pi}((-1)^n - 1) - i(-1)^n\right]$$

This will allow us to do integrals like

$$\begin{aligned} X_{nm} &= \int_0^L dx\, x \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) \\ &= \int_0^L dx\, x\left[\frac{1}{2i}\left(e^{in\pi x/L} - e^{-in\pi x/L}\right)\right]\left[\frac{1}{2i}\left(e^{im\pi x/L} - e^{-im\pi x/L}\right)\right] \\ &= -\frac{1}{4}\int_0^L dx\, x\left(e^{i(n+m)\pi x/L} - e^{i(n-m)\pi x/L} - e^{i(m-n)\pi x/L} + e^{-i(n+m)\pi x/L}\right) \end{aligned}$$

If $n = m$, $X_{nn} = L^2/4$. If $n + m$ is even then so is $n - m$, and we get $X_{nm} = 0$. If $n + m$ is odd, then

$$X_{nm} = \frac{L^2}{\pi^2}\left(\frac{1}{(n+m)^2} - \frac{1}{(n-m)^2}\right)$$

You have a particle in a 1D box with $0 \le x \le L$, in the same state Ψ as in Assignment 10:

$$\Psi(x,t) = \frac{1}{\sqrt{2}}\left[\psi_1(x)e^{-i\omega_1 t} + i\psi_2(x)e^{-i\omega_2 t}\right]$$

1. Find $\langle x\rangle$, the expectation (average) value of position in this state:

$$\langle x\rangle = \int_0^L dx\, \Psi^* x \Psi$$

Answer: Writing it out,

$$\begin{aligned}
\langle x \rangle &= \frac{1}{2}\frac{2}{L}\int_0^L dx\, x \left[\sin^2\left(\frac{\pi x}{L}\right) + \sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{2\pi x}{L}\right)\left(ie^{-i(\omega_2-\omega_1)t} - ie^{i(\omega_2-\omega_1)t}\right) + \sin^2\left(\frac{2\pi x}{L}\right)\right] \\
&= \frac{1}{L}\left[X_{11} - X_{12} i(2i)\sin(\omega_2-\omega_1)t + X_{22}\right] = \frac{1}{L}\left[\frac{L^2}{4} - 2\frac{8L^2}{9\pi^2}\sin(\omega_2-\omega_1)t + \frac{L^2}{4}\right] \\
&= L\left[\frac{1}{2} - \frac{16}{9\pi^2}\sin\left(\frac{3\hbar\pi^2}{2mL^2}t\right)\right]
\end{aligned}$$

2. Your result should remind you of a classical particle bouncing back and forth between the boundaries of the box. What is the frequency of this motion?

Answer: As seen above, the angular frequency is

$$\omega = \omega_2 - \omega_1 = \frac{3\hbar\pi^2}{2mL^2}$$