PHYS 191 Exam 3 Solutions

1. (50 points) The Schrödinger equation for a free particle in 2D is

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2}\right)$$

(a) Find out whether this equation is invariant under rotations, such that

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2\Psi}{\partial x'^2} + \frac{\partial^2\Psi}{\partial y'^2}\right)$$

Remember that

$$\frac{\partial\Psi}{\partial x} = \frac{\partial x'}{\partial x}\frac{\partial\Psi}{\partial x'} + \frac{\partial y'}{\partial x}\frac{\partial\Psi}{\partial y'}$$

Answer: Using $x' = x\cos\theta + y\sin\theta$ and $y' = y\cos\theta - x\sin\theta$,

$$\frac{\partial x'}{\partial x} = \cos\theta \quad \text{and} \quad \frac{\partial y'}{\partial x} = -\sin\theta \quad \Rightarrow \quad \frac{\partial\Psi}{\partial x} = \cos\theta\,\frac{\partial\Psi}{\partial x'} - \sin\theta\,\frac{\partial\Psi}{\partial y'}$$

Doing the derivative twice,

$$\frac{\partial^2\Psi}{\partial x^2} = \cos^2\theta\,\frac{\partial^2\Psi}{\partial x'^2} - 2\sin\theta\cos\theta\,\frac{\partial^2\Psi}{\partial x'\partial y'} + \sin^2\theta\,\frac{\partial^2\Psi}{\partial y'^2}$$

Next,

$$\frac{\partial x'}{\partial y} = \sin\theta \quad \text{and} \quad \frac{\partial y'}{\partial y} = \cos\theta \quad \Rightarrow \quad \frac{\partial\Psi}{\partial y} = \sin\theta\,\frac{\partial\Psi}{\partial x'} + \cos\theta\,\frac{\partial\Psi}{\partial y'}$$

$$\frac{\partial^2\Psi}{\partial y^2} = \sin^2\theta\,\frac{\partial^2\Psi}{\partial x'^2} + 2\sin\theta\cos\theta\,\frac{\partial^2\Psi}{\partial x'\partial y'} + \cos^2\theta\,\frac{\partial^2\Psi}{\partial y'^2}$$

When we add the second derivatives, the mixed partial derivative terms will cancel out, and

$$\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} = (\sin^2\theta + \cos^2\theta)\left(\frac{\partial^2\Psi}{\partial x'^2} + \frac{\partial^2\Psi}{\partial y'^2}\right) = \frac{\partial^2\Psi}{\partial x'^2} + \frac{\partial^2\Psi}{\partial y'^2}$$

Therefore, the Schrödinger equation for a free particle is invariant under rotations.

(b) Find out whether this equation is Lorentz invariant, such that

$$i\hbar\frac{\partial\Psi}{\partial t'} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2\Psi}{\partial x'^2} + \frac{\partial^2\Psi}{\partial y^2}\right)$$

Remember that

$$\frac{\partial \Psi}{\partial t} = c\frac{\partial \Psi}{\partial (ct)}$$

Answer: This is obviously not going to work, because the equation has second space derivatives and the first time derivative. Using $ct' = \gamma ct - \beta\gamma x$ and $x' = -\beta\gamma ct + \gamma x$,

$$\frac{\partial \Psi}{\partial t} = c\frac{\partial \Psi}{\partial (ct)} = c\frac{\partial x'}{\partial (ct)}\frac{\partial \Psi}{\partial x'} + c\frac{\partial (ct)'}{\partial (ct)}\frac{\partial \Psi}{\partial (ct)'}$$

$$\frac{\partial x'}{\partial (ct)} = -\beta\gamma \quad \text{and} \quad \frac{\partial (ct)'}{\partial (ct)} = \gamma \quad \Rightarrow \quad \frac{\partial \Psi}{\partial t} = -c\beta\gamma\,\frac{\partial \Psi}{\partial x'} + c\gamma\,\frac{\partial \Psi}{\partial (ct)'}$$

Now the x-derivatives:

$$\frac{\partial \Psi}{\partial x} = \frac{\partial x'}{\partial x}\frac{\partial \Psi}{\partial x'} + \frac{\partial (ct)'}{\partial x}\frac{\partial \Psi}{\partial (ct)'}$$

$$\frac{\partial x'}{\partial x} = \gamma \quad \text{and} \quad \frac{\partial (ct)'}{\partial x} = -\beta\gamma \quad \Rightarrow \quad \frac{\partial \Psi}{\partial x} = \gamma\,\frac{\partial \Psi}{\partial x'} - \beta\gamma\,\frac{\partial \Psi}{\partial (ct)'}$$

$$\frac{\partial^2 \Psi}{\partial x^2} = \gamma^2\,\frac{\partial^2 \Psi}{\partial x'^2} - 2\beta\gamma^2\,\frac{\partial^2 \Psi}{\partial x'\partial (ct)'} + \beta^2\gamma^2\,\frac{\partial^2 \Psi}{\partial (ct)'^2}$$

The overall equation ends up as

$$i\hbar\left(-c\beta\gamma\,\frac{\partial \Psi}{\partial x'} + c\gamma\,\frac{\partial \Psi}{\partial (ct)'}\right) = -\frac{\hbar^2}{2m}\left(\gamma^2\,\frac{\partial^2 \Psi}{\partial x'^2} - 2\beta\gamma^2\,\frac{\partial^2 \Psi}{\partial x'\partial (ct)'} + \beta^2\gamma^2\,\frac{\partial^2 \Psi}{\partial (ct)'^2} + \frac{\partial^2 \Psi}{\partial y^2}\right)$$

There are no cancellations. The Schrödinger equation for a free particle is *not* Lorentz invariant!

2. (50 points) You have a particle in a 1D box with $0 \le x \le L$, in the same state Ψ you had in Assignment 10:

$$\Psi(x,t) = \frac{1}{\sqrt{2}}\left[\psi_1(x)e^{-i\omega_1 t} + i\psi_2(x)e^{-i\omega_2 t}\right]$$

(a) Find $\langle p \rangle$, the expectation (average) value of momentum in this state:

$$\langle p \rangle = \int_0^L dx\;\Psi^*\hat{p}\Psi = -i\hbar\int_0^L dx\;\Psi^*\frac{\partial \Psi}{\partial x}$$

Hint: Start with the known integral

$$I(n\pi) = \int_0^L dx\;e^{in\pi x/L}$$

That's all the integration you will need to do.

Answer: We know that $I(n\pi) = ((-1)^n - 1)L/in\pi$, which is L for $n = 0$, and otherwise zero for even n and $-2L/in\pi$ for odd n. The integrals we need to perform will be

$$\begin{aligned}
P_{nm} &= \int_0^L dx \, \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) \\
&= \int_0^L dx \, \frac{1}{2i}\left(e^{in\pi x/L} - e^{-in\pi x/L}\right)\frac{1}{2}\left(e^{im\pi x/L} + e^{-im\pi x/L}\right) \\
&= \frac{1}{4i}\int_0^L dx \, \left(e^{i(n+m)\pi x/L} + e^{i(n-m)\pi x/L} - e^{i(m-n)\pi x/L} - e^{-i(n+m)\pi x/L}\right) \\
&= \frac{1}{4i}\left[I\left((n+m)\pi\right) + I\left((n-m)\pi\right) - I\left((m-n)\pi\right) - I\left(-(n+m)\pi\right)\right]
\end{aligned}$$

For $n \pm m$ even, $P_{nm} = 0$. For $n \pm m$ odd,

$$P_{nm} = \frac{L}{\pi}\left[\frac{1}{n+m} + \frac{1}{n-m}\right]$$

Now for our expectation value:

$$\begin{aligned}
\langle p \rangle &= -\frac{i\hbar}{2}\frac{2}{L}\int_0^L dx \, \left[\sin\left(\frac{\pi x}{L}\right)e^{i\omega_1 t} - i\sin\left(\frac{2\pi x}{L}\right)e^{i\omega_2 t}\right] \\
&\qquad \left[\frac{\pi}{L}\cos\left(\frac{\pi x}{L}\right)e^{-i\omega_1 t} + i\frac{2\pi}{L}\cos\left(\frac{2\pi x}{L}\right)e^{-i\omega_2 t}\right] \\
&= -\frac{i\pi\hbar}{L^2}\left[P_{11} + 2ie^{-i(\omega_2-\omega_1)t}P_{12} - ie^{i(\omega_2-\omega_1)t}P_{21} + 2P_{22}\right] \\
&= \frac{\hbar}{L}\left[2e^{-i(\omega_2-\omega_1)t}\left(\frac{1}{3} - 1\right) - e^{i(\omega_2-\omega_1)t}\left(\frac{1}{3} + 1\right)\right] \\
&= -\frac{4}{3}\frac{\hbar}{L}\left(e^{i(\omega_2-\omega_1)t} + e^{-i(\omega_2-\omega_1)t}\right) = -\frac{8\hbar}{3L}\cos\left(\frac{3\hbar\pi^2}{2mL^2}t\right)
\end{aligned}$$

(b) For a classical particle, $p = m\frac{d}{dt}x$. You can't do that for a quantum particle. But you can check if $\langle p \rangle = m\frac{d}{dt}\langle x \rangle$. Is this so?

Answer: Checking:

$$mL\frac{d}{dt}\left[\frac{1}{2} - \frac{16}{9\pi^2}\sin\left(\frac{3\hbar\pi^2}{2mL^2}t\right)\right] = -\frac{16mL}{9\pi^2}\frac{3\hbar\pi^2}{2mL^2}\cos\left(\frac{3\hbar\pi^2}{2mL^2}t\right) = -\frac{8\hbar}{3L}\cos\left(\frac{3\hbar\pi^2}{2mL^2}t\right)$$

So it works!