1. (40 points) You have a flat universe that contains matter and dark energy with $w_d = -\frac{2}{3}$ and $\Omega_{d,0} = 0.7$.

(a) What was $\varepsilon_d/\varepsilon_m$ at the time when light with $z = 1$ was emitted from a distant galaxy?

Answer: Since $1 + z = 1/a(t_e)$, at the time when $z = 1$, the scale factor must have been $a = \frac{1}{2}$. For matter, $\varepsilon_m = \varepsilon_{m,0}a^{-3}$; for dark energy, $\varepsilon_d = \varepsilon_{d,0}a^{-3(1+w_d)} = \varepsilon_{d,0}a^{-1}$. Therefore,

$$\frac{\varepsilon_d}{\varepsilon_m} = \frac{\varepsilon_{d,0}}{\varepsilon_{m,0}}a^2 = \frac{\Omega_{d,0}}{\Omega_{m,0}}a^2 = \frac{\Omega_{d,0}}{(1-\Omega_{d,0})}a^2 = \frac{7}{12}$$

(b) Find a value for the scale factor, $\bar{a}$, such that when $a \gg \bar{a}$, the dark energy will completely dominate the evolution of the scale factor. *Hint:* Look for a value for a (a numerical value) where the matter and dark energy terms in the Friedmann equation are comparable to one another.

Answer: The Friedmann equation gives the time evolution of a:

$$\frac{H^2}{H_0^2} = \frac{(1-\Omega_{d,0})}{a^3} + \frac{\Omega_{d,0}}{a}$$

At the $\bar{a}$ when $\varepsilon_m = \varepsilon_d$, the matter and dark energy terms in the Friedmann equation will be equal. When $a \gg \bar{a}$, the dark energy will dominate.

$$\frac{\varepsilon_d}{\varepsilon_m} = \frac{\Omega_{d,0}}{(1-\Omega_{d,0})}\bar{a}^2 = 1 \quad \Rightarrow \quad \bar{a} = \sqrt{3/7}$$

Any $\bar{a} \approx 1$ will work, since the limit where $a \gg \bar{a}$ is what matters.

(c) In the limit where $a \gg \bar{a}$, find the time-dependence of the scale factor, and compare it to the late-times $a(t)$ for a universe with matter and a cosmological constant. Would you expect the late universe with this dark energy characterized by $w_d = -\frac{2}{3}$ to expand faster or slower than the universe with a cosmological constant ? Why?

Answer: Late-times Friedmann equation:

$$\frac{H^2}{H_0^2} = \frac{\Omega_{d,0}}{a} \quad \Rightarrow \quad \dot{a} = H_0\sqrt{\Omega_{d,0}}\, a^{1/2}$$

Integrating (but leaving the lower limits unspecified), we find

$$\int^a \frac{da'}{\sqrt{a'}} = H_0\sqrt{\Omega_{d,0}} \int^t dt' \quad \Rightarrow \quad 2\sqrt{a} = H_0\sqrt{\Omega_{d,0}}t + \text{const}$$

The result is

$$a = [0.418 H_0\, t + \text{const}]^2$$

This $a \sim t^2$ growth is slower than the exponential growth produced by a cosmological constant. This is because this form of dark energy has an energy density that decreases (like a^{-1}) as the universe expands, while the energy density due to a cosmological constant remains constant.

2. (50 points) Find the requirement that w_e of the dominant form of energy in the early universe must meet for there to be a finite horizon distance.

(a) Start with equation (5.51), but write down a form that applies to all times, not just the present. Then show that this is the distance a photon will travel to get to the origin in a universe described by a Robertson-Walker metric.

Answer: Using the Robertson-Walker metric, for a photon, $ds = 0$. If it's traveling toward the origin, $d\Omega = 0$. So we're left with

$$0 = -c^2 dt^2 + a^2 dr^2 \quad \Rightarrow \quad c\, dt = a\, dr$$

Integrating this for a photon traveling since $t = 0$, we get the horizon distance

$$d_{\text{hor}}(t) = c\int_0^t \frac{dt'}{a}$$

(b) Write down the Friedmann equation for a flat, very early universe where one component with the largest $w = w_e$ will dominate. Use this (not the process leading to equation 5.52) to find under what conditions the integral for the horizon distance diverges or converges. Does this result apply to non-flat universes as well?

Answer: With the dominant term for $a \ll 1$ only,

$$\frac{H^2}{H_0^2} = \frac{\Omega_{e,0}}{a^{3(1+w_e)}} \quad \Rightarrow \quad \dot{a} \propto a^{-(1+3w_e)/2} \ \text{ and } \ dt \propto da\, a^{(1+3w_e)/2}$$

Here, I've dropped the overall multiplying constants and used proportionality, since all we care about is convergence or divergence of the d_{hor} integral.

$$d_{\text{hor}} \propto \int_0^a da\, a^{(1+3w_e)/2-1} = \left. \frac{a^{(1+3w_e)/2}}{(1+3w_e)/2} \right|_0^a$$

This integral diverges at the lower limit unless $(1+3w_e)/2 > 0$, or $w_e > -\frac{1}{3}$.

With non-flat universes, we have

$$\frac{H^2}{H_0^2} = \frac{\Omega_{e,0}}{a^{3(1+w_e)}} + \frac{1-\Omega_0}{a^2}$$

For $w_e > -\frac{1}{3}$, the first term will still be dominant as $a \to 0$, so the argument will not change. For $w_e \leq -\frac{1}{3}$, the curvature will dominate, and we will get something like an empty universe with $\kappa = -1$, where $a \propto t$. In that case,

$$d_{\text{hor}} \propto \int_0^t \frac{dt}{t} \to \infty$$

So again, it makes no difference.

(c) Can you generalize your result for the early universe to the present or not? Explain.

Answer: The integral for d_{hor} can diverge only at the lower limit, unless $a = 0$ for any finite value of t. If you have a model with a Big Crunch and you try to get d_{hor} for a time *after* the Big Crunch, this could happen. But that value would not be meaningful. Otherwise, if you have a finite d_{hor} for an arbitrarily small time, it will always remain finite, and if it's infinite, it will always remain so.

3. (10 points) A friend tells you that she understands that the force of gravity acts to slow the expansion of the universe, and that there might be a force due to a cosmological constant that accelerates the expansion. But, she asks, what is the force that makes the universe expand from the Big Bang in the first place? Is it the cosmological constant? How would you answer her?

Answer: There are a number of possible conceptual confusions here.

- If she is thinking of a force driving the expansion, this is similar to a naive student thinking you need a force acting on an object that is tossed upward. No—the upward movement is an initial condition, and the only force is gravity, which produces a downward acceleration. The upward movement is simply one possible solution to the equations of motion, fixed by the initial conditions. Similarly, a solution to Friedmann's equation with $\dot{a} > 0$ does not require a force acting on the universe to make it expand.

- To make things more difficult, in general relativity, you should not think of forces in a Newtonian sense. All "forces" in GR are analogous to "fictitious forces" that appear when you go to non-inertial frames of reference. All that we have is curvature and acceleration, and no intermediating concept of "force" is necessary.

- If she is really asking what *caused* the Big Bang, you just have to say physicists don't really know, at least not without a decent theory of quantum gravity. But then, you have to add that just like GR complicates the everyday notion of a force, quantum mechanics doesn't work well with naive, everyday notions of causality, due to its inherent randomness.

In other words, the original question is not well-formed, depending too much on everyday, intuitive notions of force or causality that break down in extreme physical contexts such as cosmology.