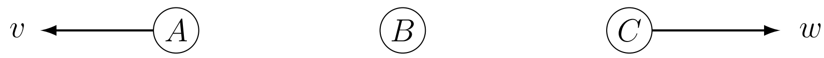
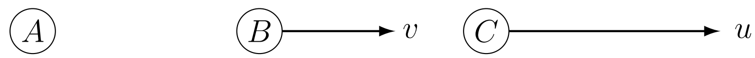
Solutions to Exam 3; Phys 186

1. (30 points) You have three objects, A, B, C. In the frame of reference where B is stationary, this is how the velocities of A and C look. The speeds are v and w.



Draw in how the velocities look in the reference frame where A is stationary. Call the speed of C in this frame u.



Now, write down the value of the speed u when $w = 0$, and provide a short (one sentence) explanation for your answer.

$u = v$ C will be in the same reference frame as B, moving at v.

Write down the value of the speed u when $w = c$, and provide a short explanation.

$u = c$ The speed of light is the same in all reference frames.

Finally, use what you have done so far to pick the correct answer for u among the five options given. Include your calculations and explain your reasoning.

(a) $u = v + w$

Answer: Wrong. For $w = 0$ this gives $u = v$, which is correct. But for $w = c$, this gives $u = v + c$, which is wrong.

(b) $u = \frac{v+w}{\sqrt{1-v^2/c^2}}$

Answer: Wrong. For $w = 0$ this gives $u = v/\sqrt{1 - v^2/c^2}$, which is wrong. For $w = c$, this gives $u = (v + c/)\sqrt{1 - v^2/c^2}$, which is also wrong.

(c) $u = (v + w)\sqrt{1 - w^2/c^2}$

Answer: Wrong. For $w = 0$ this gives $u = v$, which is correct. But for $w = c$, this gives $u = 0$, which is wrong.

(d) $u = \frac{v+w}{1+vw/c^2}$

Answer: Correct! For $w = 0$ this gives $u = v$, which is correct. For $w = c$, this gives $u = c$, which is also correct.

(e) $u = (v + w)(1 + wv/c^2)$

Answer: Wrong. For $w = 0$ this gives $u = v$, which is correct. But for $w = c$, this gives $u = (1 + v/c)^2 c$, which is wrong.

2. (35 points) A black hole has entropy $S = \kappa A$, where κ is a constant and A is the surface area of the event horizon of the black hole.

(a) Find equations giving the entropy of a black hole of mass 0, M, $2M$, and $3M$.

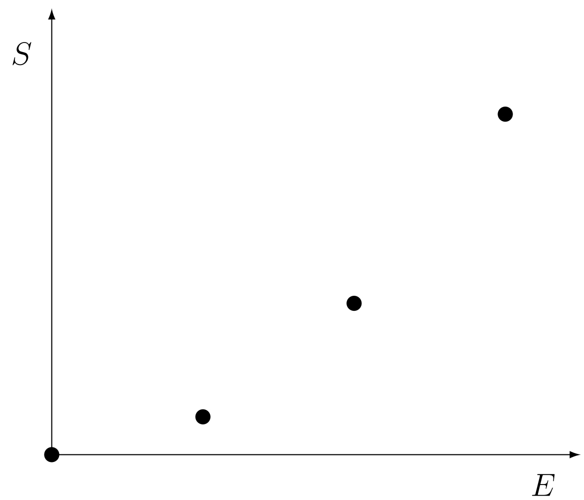
Answer: The surface area of a black hole with mass m is

$$A = 4\pi R^2 = 4\pi \left(\frac{2Gm}{c^2}\right)^2 = \left(\frac{16\pi G^2}{c^4}\right) m^2$$

For the values given, then,

$$S = 0, \quad \kappa \left(\frac{16\pi G^2}{c^4}\right) M^2, \quad 2\kappa \left(\frac{16\pi G^2}{c^4}\right) M^2, \quad 9\kappa \left(\frac{16\pi G^2}{c^4}\right) M^2$$

(b) Use your previous result to sketch the graph of how the entropy of a black hole depends on its *total energy* E. Explain your reasoning.



Answer: The total energy is $E = mc^2$, and $m = E/c^2$. The entropy is therefore

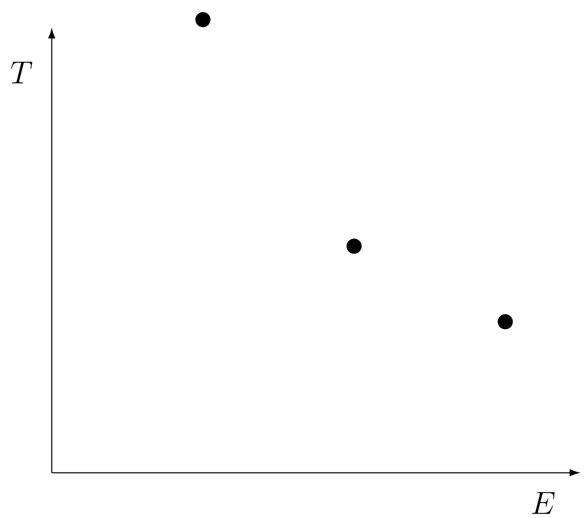
$$S = \kappa \left(\frac{16\pi G^2}{c^8} \right) E^2$$

In other words, $S \propto E^2$ and the graph is a parabola.

(c) To define the temperature of an object properly, we use

$$\frac{1}{T} = \frac{dS}{dE}$$

In other words, the inverse of the absolute temperature is the rate of change of its entropy with respect to its total energy. Therefore you can use the previous graph to make a rough sketch of how the temperature of a black hole depends on its total energy.



Answer: Graphically, the rate of change corresponds to the slope of the graph. With a parabola, the slope starts from zero at $E = 0$, and then becomes increasingly positive as E increases. The slope, however, is $1/T$, not the temperature. The temperature graph therefore starts at $+\infty$ at $E = 0$, is positive, and *decreases* as E increases.

You will notice this in the equation for the temperature you used in Assignment 8 question 3, where $T \propto 1/M$.

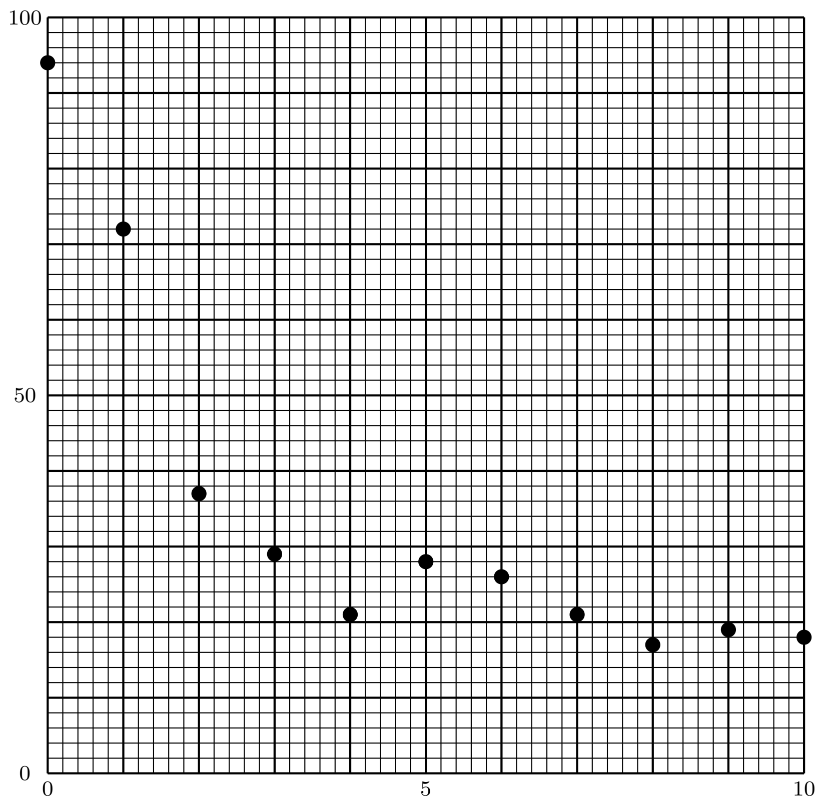
(d) The specific heat of an object tells you by how much the temperature of an object increases if you transfer a small amount of energy to it in the form of heat. Is the specific heat of a black hole positive or negative? Are the specific heats of everyday objects positive or negative? Explain your reasoning. *Hint:* Look at your graph. If you increase the energy by a small amount, will the temperature go up or go down?

Answer: Since the temperature goes down as you add energy (mass) to a black hole, it has a negative specific heat. With ordinary objects, the temperature goes up as you transfer energy (heat) to it, so specific heats are positive. Black holes are strange.

3. (35 points) You have a radioactive sample. Every day at exactly the same time, you take a Geiger counter and count the total number of events with your sample in place for exactly one minute. Call this activity A_T. Here is a table of your data:

t (days)	0	1	2	3	4	5	6	7	8	9	10
A_T (counts/minute)	94	72	37	29	21	28	26	21	17	19	18

(a) Make a graph of A_T, with time on the horizontal axis.

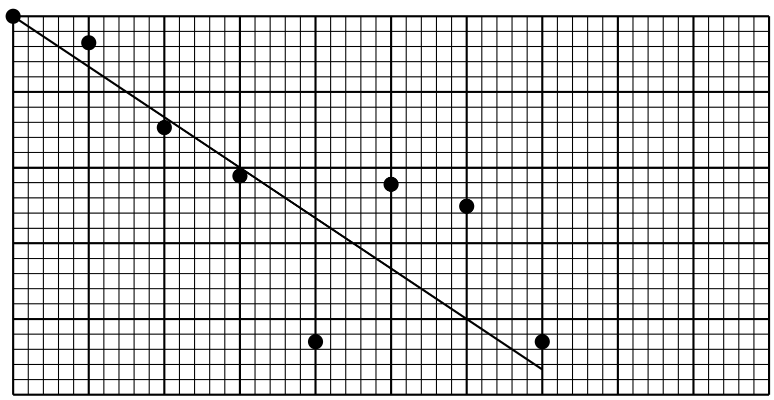


Estimate the background radioactivity for this case. Explain your reasoning.

Answer: The data is noisy, because radioactive decay events are random, and with $N \approx 100$, the standard deviation $\approx \sqrt{100} = 10$. But it looks like the graph levels off around 20 counts/minute. After many half-lives pass, the radioactivity will reach background. So about 20 is a good estimate for the background.

(b) Make a graph of $\ln(A_S/A_{S0})$ versus t, where A_S is the radioactivity due to your sample alone, and A_{S0} is the radioactivity from the sample alone at $t = 0$.

t (days)	0	1	2	3	4	5	6	7	8	9	10
A_S/A_{S0}	1	0.70	0.23	0.12	0.01	0.11	0.08	0.01	-0.04	-0.01	-0.02
$\ln(A_S/A_{S0})$	0	-0.35	-1.47	-2.11	-4.3	-2.22	-2.51	-4.3	—	—	—



Estimate the half-life of your sample. Provide your calculations.

Answer: If you draw a line through the data points, you'll get a slope of around -0.6 to -0.7 in units of 1/days. The line I drew above has a slope of $-\frac{2}{3}$/day, so say -0.67/day. The exponential decay associated with half-lives is

$$A_S = A_{S0}\, e^{-t \ln 2/t_{1/2}} \qquad \Rightarrow \qquad \ln\left(\frac{A_S}{A_{S0}}\right) = -\frac{\ln 2}{t_{1/2}} t$$

This means the slope of the line is

$$-0.67/\text{day} = -\frac{\ln 2}{t_{1/2}} \qquad \Rightarrow \qquad t_{1/2} = \frac{\ln 2}{0.65} = 1.03\,\text{days}$$

I generated the data by using a half-life of 1 day, so that is a decent estimate.