PHYS 191 Exam 1 Solutions

In 3D, spherical waves radiating away from a point source cannot be described by sines and cosines. The equation for waves that depend only on r, the radial distance to the source, and not on ϕ and θ, the spherical angle coordinates, is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f(r,t)}{\partial r}\right) = \frac{1}{v^2}\frac{\partial^2 f(r,t)}{\partial t^2}$$

where $f(r,t)$ is the wave, and v is the speed of the wave.

1. (20 points) Show that if you try a solution like $f = Ae^{i(kr-\omega t)}$, it does *not* solve the 3D radial wave equation (the equation above) for $k \neq 0$, $\omega \neq 0$.

Answer: The left hand side:

$$\begin{aligned}
\frac{A}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}e^{i(kr-\omega t)}\right) &= \frac{A}{r^2}\frac{\partial}{\partial r}\left(ikr^2 e^{i(kr-\omega t)}\right) \\
&= Ae^{i(kr-\omega t)}\left(\frac{2ik}{r} - k^2\right)
\end{aligned}$$

This can't be made equal to the right hand side

$$\begin{aligned}
\frac{A}{v^2}\frac{\partial^2 e^{i(kr-\omega t)}}{\partial t^2} &= \frac{A}{v^2}\frac{\partial}{\partial t}\left(-i\omega e^{i(kr-\omega t)}\right) \\
&= -\frac{A}{v^2}\omega^2 e^{i(kr-\omega t)}
\end{aligned}$$

The only way to make these equal is for $k = 0$ and $\omega = 0$ for $f = A$, which is not an interesting solution.

2. (40 points) Now try $f = \frac{A}{r}e^{i(kr-\omega t)}$. What must the relationship between ω and k be to satisfy the 3D radial wave equation?

Answer: The left hand side:

$$\begin{aligned}
\frac{A}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}\left(\frac{e^{i(kr-\omega t)}}{r}\right)\right] &= \frac{A}{r^2}\frac{\partial}{\partial r}\left[r^2\left(\frac{ik}{r}e^{i(kr-\omega t)} - \frac{1}{r^2}e^{i(kr-\omega t)}\right)\right] \\
&= \frac{A}{r^2}\frac{\partial}{\partial r}\left[ikre^{i(kr-\omega t)} - e^{i(kr-\omega t)}\right] \\
&= \frac{A}{r^2}\left(ike^{i(kr-\omega t)} - k^2re^{i(kr-\omega t)} - ike^{i(kr-\omega t)}\right) \\
&= -\frac{k^2A}{r}e^{i(kr-\omega t)}
\end{aligned}$$

The right hand side:

$$\begin{aligned} \frac{A}{v^2}\frac{\partial^2}{\partial t^2}\left(\frac{e^{i(kr-\omega t)}}{r}\right) &= \frac{A}{v^2 r}\frac{\partial}{\partial t}\left(-i\omega e^{i(kr-\omega t)}\right) \\ &= -\frac{A}{v^2 r}\omega^2 e^{i(kr-\omega t)} \end{aligned}$$

These will be equal if $k^2 = \omega^2/v^2$ or $v = \omega/k$.

3. (10 points) The solution you've tried for f is complex-valued. Given your experience with 1D waves, propose a real-valued solution for $f(r,t)$. If you're confident enough in your proposal, you don't have to show that it satisfies the wave equation. Remember that A above is a complex number!

Answer: Just as with 1D waves, take the real part of your solution:

$$f(r,t) = \mathrm{Re}\left[\frac{A}{r}e^{i(kr-\omega t)}\right] = \frac{|A|}{r}\cos(kr - \omega t + \delta)$$

4. (10 points) At a fixed location in space (and therefore a fixed radius r), $f(r,t)$ will oscillate in time like a simple harmonic oscillator. What will the amplitude of such oscillations be?

Answer: The real solution will vary between $\pm|A|/r$. So the amplitude is $|A|/r$.

5. (20 points) In a simple harmonic oscillator, the total energy is proportional to the amplitude squared: $E_T \propto |A|^2$. A wave is a traveling oscillation, so the energy transmitted by a wave through a location is proportional to the square of the amplitude of the wave at that location. Spherical 3D waves from a point source spread out in concentric spheres, distributing their energy over the area of those spheres. That is, the energy transmitted through any point a distance r from the source is proportional to the amplitude of the wave at r, therefore, the total energy transmitted over a sphere with radius r is

$$E_{\text{sphere}} \propto \int_{\text{sphere}} da\,|\text{amplitude}|^2$$

Here, da refers to an infinitesimal area element, just as in Activity 5.

If energy is conserved for spherical 3D waves, the total energy transmitted through a sphere centered on the source should not depend on r. Is that so?

Answer: Do the integral:

$$E_{\text{sphere}} \propto \int_{\text{sphere}} da\left|\frac{A}{r}\right|^2 = \frac{|A|^2}{r^2}\int_{\text{sphere}} da = \frac{|A|^2}{r^2}4\pi r^2 = \text{const}$$

The result does not depend on r; energy is conserved.