Phys 191 Activity 4: Electric Fields

$$\vec{E} = \sum_i \frac{kq_i}{r_i^2}\hat{\mathbf{r}}_i = k\sum_i q_i \left(\frac{\hat{\mathbf{r}}_i}{r_i^2}\right) \qquad \longrightarrow \qquad \vec{E} = \int d\vec{E} = k\int dq \left(\frac{\hat{\mathbf{r}}}{r^2}\right)$$

1. A circle of radius R lies in the xy plane and carries a uniformly distributed charge Q. Find the electric field along the z axis.

(a) You know the electric field due to a point charge. So a good dq would be an infinitesimal bit of charge at a point on the circle. Take dq to be located at $y = 0$ and $x = R$. Write down dE_x, dE_y, and dE_z, the components of the electric field produced by this dq a distance z along the z-axis. Start by drawing a picture; it will help you do the trigonometry.

Answer: You have a triangle with r as the hypotenuse, with r the distance between the charge dq and the point on the z-axis. The magnitude of the electric field for a point charge gives

$$dE = k\frac{dq}{r^2} = k\frac{dq}{z^2+R^2}$$

On the z-axis, we also have

$$\hat{\mathbf{r}} = -\frac{R}{r}\hat{\mathbf{x}} + \frac{z}{r}\hat{\mathbf{z}}$$

Since $d\vec{E} = dE\,\hat{\mathbf{r}}$, this means that

$$dE_x = -k\frac{dq\,R}{(z^2+R^2)^{3/2}}, \quad dE_y = 0, \quad dE_z = k\frac{dq\,z}{(z^2+R^2)^{3/2}}$$

(b) The electric field produced by dq is $d\vec{E} = dE_x\hat{\mathbf{x}} + dE_y\hat{\mathbf{y}} + dE_z\hat{\mathbf{z}}$. When you add all the contributions from all the dq's, you will get the electric field:

$$E_x\hat{\mathbf{x}} = \int_{\text{ring}} dE_x\,\hat{\mathbf{x}} = \hat{\mathbf{x}}\int_{\text{ring}} dE_x \quad \text{and} \quad E_y\hat{\mathbf{y}} = \ldots, \quad E_z\hat{\mathbf{z}} = \ldots$$

Due to the symmetry of the situation, you will have to do only one of these integrals; the other two electric field components will be zero. Which one will you actually need to calculate: E_x, E_y, or E_z?

Answer: Only E_z. All contributions from all dq along the ring will have an identical dE_z, so all those will add up to something non-zero. But every x or y-component of an electric field produced by a dq will be canceled out by an equal and opposite electric field component from another dq located on the exact opposite side of the circle. Due to the symmetry of the situation, only E_z survives.

(c) Integrating over the ring is easier in polar coordinates: r and ϕ rather than x and y. Now, dq will be the share of charge distributed over $d\phi$, when the full charge Q is evenly spread over a full circle with angle 2π. Write down what dq is, and write what $\int_{\text{ring}} dq$ is in polar coordinates.

Answer: The full circle is 2π radians. Each infinitesimal angle has $Q\,d\phi/2\pi$ of the total charge. Therefore

$$dq = Q\frac{d\phi}{2\pi} \qquad \text{and} \qquad \int_{\text{ring}} dq = \frac{Q}{2\pi}\int_0^{2\pi} d\phi$$

(d) Do the integral, and find the electric field along the z axis. What happens as $z \gg R$?

Answer: We only need to get E_z:

$$E_z = \int_{\text{ring}} dE_z = k\int_{\text{ring}} \frac{dq\, z}{(z^2+R^2)^{3/2}} = k\frac{Q}{2\pi}\frac{z}{(z^2+R^2)^{3/2}}\int_0^{2\pi} d\phi = \frac{kQz}{(z^2+R^2)^{3/2}}$$

When very far from the ring, that is, when $z \gg R$, the ring should behave like a point charge Q. When $z \gg R$, $z^2 + R^2 \approx z^2$, therefore

$$E_z = \frac{kQz}{(z^2+R^2)^{3/2}} \to \frac{kQz}{z^3} = \frac{kQ}{z^2}$$

as it should be.

2. A line of length L lies along the x axis from $-L/2$ to $+L/2$ and carries a uniformly distributed charge Q. Find the electric field along the z axis. What happens as $z \gg L$? What if $L \to \infty$ with $Q/L = \lambda = \text{constant}$?

Answer: Again, due to symmetry, we only have to do E_z. Take a charge $dq = Q\,dx/L$ located at x; that produces $dE_z = k\,dq\,z/(z^2+x^2)^{3/2}$. Integrating this:

$$E_z = \int_{\text{line}} dE_z = \frac{kQz}{L}\int_{-L/2}^{L/2}\frac{dx}{(z^2+x^2)^{3/2}} = \frac{kQz}{L}\left.\frac{x}{z^2(z^2+x^2)^{1/2}}\right|_{-L/2}^{L/2} = \frac{kQ}{z\left[z^2+\left(\frac{L}{2}\right)^2\right]^{1/2}}$$

As $z \gg L$, the line behaves like a point charge; $\left[z^2+\left(\frac{L}{2}\right)^2\right]^{1/2} \approx z$, so $E_z \to kQ/z^2$.

As $L \to \infty$, $\left[z^2+\left(\frac{L}{2}\right)^2\right]^{1/2} \to \frac{L}{2}$. In that case, $E_z \to 2k\lambda/z$, which you may have encountered before as the electric field due to an infinitely long straight wire.